INTRODUCTION TO POLAR COORDINATES.

Q. If the gravitational attraction of Sun is pulling Earth radially inwards, why Earth does not collapse onto the Sun. Why does Earth rotate in an elliptical orbit, ~~tangential~~ normal to the radially inward force?

Ans. • Why do we expect Earth to collapse on to Sun in the first place? Because as we push the duster on the table it moves in the direction of force.

• Now why shouldn't we expect things to not necessarily displace in the direction of force?

Because $\vec{F} \propto \vec{a} = \frac{d^2\vec{x}}{dt^2}$.

Force is proportional to accelaration which is second derivative of displacement.

• Thus if we explore the meaning of the derivative of a vector, it will clarify two things

→ Under what circumstance, force is linearly related to displacement i.e., duster being pushed on table

→ Force is orthogonal to displacement i.e., motion of planet around the sun, or a mass whirled on string.

→ A more general motion which is a combination of above two possibilities.

• ~~SCALAR~~ DERIVATIVE OF A SCALAR.

Consider a scalar function of time $f(t)$. i.e., for every value of time t, it gives some number. $f(t)$ could be temperature recorded by a thermometer in your room.

df/dt then tells you instantaneous time rate of temperature and is given as

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

The temperature can either increase, decrease or stay constant with time, and df/dt is accordingly >0, <0, or $=0$, but a number nonetheless. Thus, in general a scalar can only change in magnitude with time and hence its time derivative is a scalar.

DERIVATIVE OF A VECTOR:

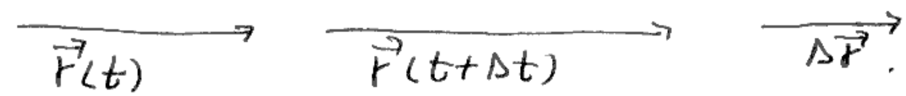
A vector has two attributes – magnitude and direction, and hence can change in time in two different and independent ways, and in general, both the ways simultaneously.

A VECTOR CHANGING PURELY IN MAGNITUDE:

When a vector changes purely in magnitude without changing its direction, its derivative is very much akin to that of a scalar. Imagine going on a highway in a straight line with a constant speed. The displacement vector $\cancel{d\vec{x}}$ $d\vec{r}$ changes only in magnitude.

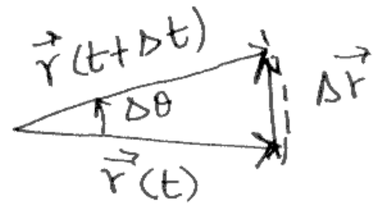
$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}.$$

Here, all the three vectors $\vec{r}(t+\Delta t)$, $\vec{r}(t)$ and $\cancel{\vec{r}(t+\Delta t)}$ $\Delta\vec{r}$, point in the same direction and hence the vector difference $\Delta\vec{r}$ is trivial to obtain.



VECTOR CHANGING PURELY IN DIRECTION

DIRECTION: The only way a vector can change in direction while staying constant in magnitude is for it to rotate keeping its tail fixed. As shown in the figure, position vector



$\vec{r}(t)$ has rotated by an angle $\Delta\theta$. Now

$$\frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

Now, unlike the previous case, $\vec{r}(t)$, $\vec{r}(t+\Delta t)$, and $\Delta\vec{r}$, all point in different direction and hence $\vec{r}(t+\Delta t)$ is obtained by vectorial addition of $\vec{r}(t)$ & $\Delta\vec{r}$.

. In the limit $\Delta t \to 0$, $\Delta\theta \to 0$, and we can approximate the magnitude $|\Delta\vec{r}| \approx |\vec{r}|\Delta\theta$. The direction of $\Delta\vec{r}$ in this limit is tangent to the arc and hence orthogonal to $\vec{r}(t)$

Thus,

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0} |\vec{r}(t)| \frac{\Delta\theta}{\Delta t} \hat{\theta}$$

$$\boxed{\frac{d\vec{r}}{dt} = |\vec{r}| \frac{d\theta}{dt} \hat{\theta}}$$

This is a very important result. It says that the time derivative of a vector that is constant in magnitude is

a) $\propto |\vec{r}|$ that is magnitude of vector

b) $\propto \frac{d\theta}{dt}$ that is angular speed because the vector is rotating

c) Is orthogonal to the initial direction of the vector.

So a vector can change in three ways

1) <u>Pure scaling</u>: Changing only in magnitude w/o changing direction

2) <u>Pure rotation</u>: Changing only in direction w/o changing magnitude.

3) A general change where it changes in both magnitude and direction.

The above discussion already hints towards the fact that in general time derivative of a vector need not point in the same direction as the vector. Thus, accelaration being second derivative of displacement, ~~has no nec~~ need not necessarily be parallel to displacement. When we are pushing a duster, the displacement is ~~pa~~ only changing in magnitude w/o changing direction. But for the motion of mass being tied to a string and whirled in a circle, the displacement stays constant in magnitude but continuously changes in direction. Thus velocity ($d\vec{r}/dt$) is orthogonal to $\vec{r}$ and, accelaration ($d\vec{v}/dt$) is orthogonal to velocity.

DECOMPOSING A VECTORIAL CHANGE INTO PURE SCALING AND PURE ROTATION LOOKS INTERESTING.

Q. Normally we resolve a 2-D vector along its x and y cartesian components.

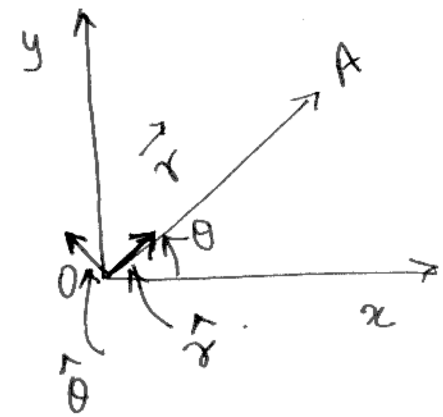
a) Do we have a coordinate system to which resolving a vectorial change into pure scaling and pure rotation is native?

b) What physical expectations shall guide the construction of such a coordinate system?

c) How different would it be from the conventional cartesian system.

<u>ANSWER</u> a) Yes, plane polar coordinates are tailor made for such a resolution

b) CONSTRUCTION OF POLAR COORDINATE SYSTEM



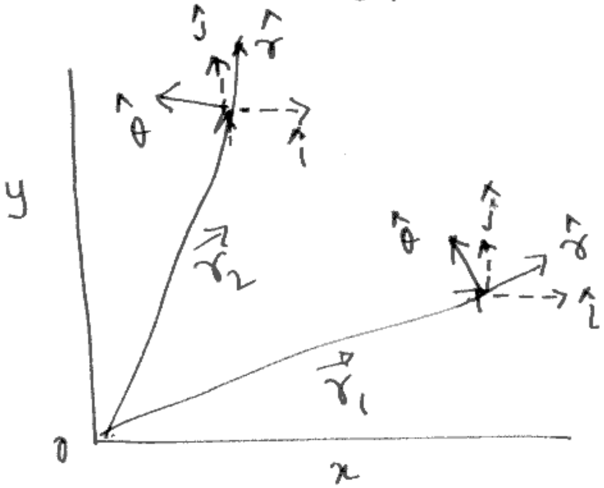
→ In 2-dim we shall require two coordinates and hence two coordinate axis.

→ Once we settle on one of the axis, the other axis choice is trivially determined by orthogonality.

→ Since it should describe the scaling of an arbitrary vector which can be pointing in an arbitrary direction, it cannot be hinged to a fixed direction like $\hat{i}$ and $\hat{j}$ of cartesian system. Multiple of a unit vector pointing in the direction of a given vector, is the only way we can describe pure scaling. Thus if line segment $\vec{OA}$ describes the length r of a particular position vector $\vec{r}$ that makes an angle θ to the x-axis, then the unit vector $\hat{r}$ pointing away from the origin and in the direction of vector $\vec{r}$ is capable of describing pure scaling of vector $\vec{r}$.

→ Having made a choice of a unit vector $\hat{r}$, the choice of other unit vector is trivial. It should be orthogonal to $\hat{r}$ and describe pure rotation of position vector $\vec{r}$ without scaling. Now let us call such a vector $\hat{\theta}$. Now we can rotate the vector clockwise or counter-clockwise. By convention we choose $\hat{\theta}$ to be positive when it describes counter-clockwise rotation and negative when it describes clockwise rotation. $\hat{r}$ is +ve radially outwards & -ve inwards.

c) POLAR COORDINATES ARE FUNDAMENTALLY DIFFERENT FROM CARTSIAN SYSTEM.



The figure on the left compares two position vectors $\vec{r}_1$ and $\vec{r}_2$ ~~in so~~ resolved in cartesian and polar coordinates.

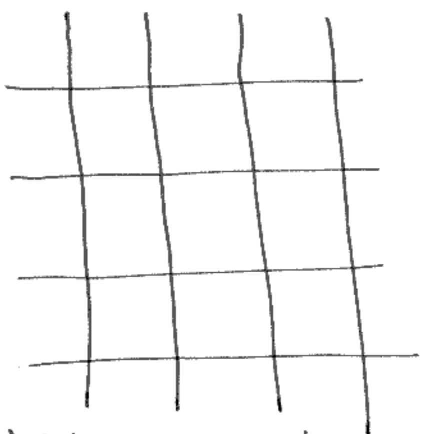
As can be clearly seen, unit vectors $\hat{i}$ and $\hat{j}$ have fixed directions once we decide upon cartesian system. Polar unit vectors $\hat{r}$ and $\hat{\theta}$ on the other hand have their directions defined by the directions of vectors they are describing.

→ Put another way cartesian unit vectors $\hat{i}$ and $\hat{j}$ are truly constant vectors. That is they are constant in both magnitude and direction. Thus their time derivative is always zero. This leads to immense simplicity in the expression for velocity and accelaration in cartesian coordinates.

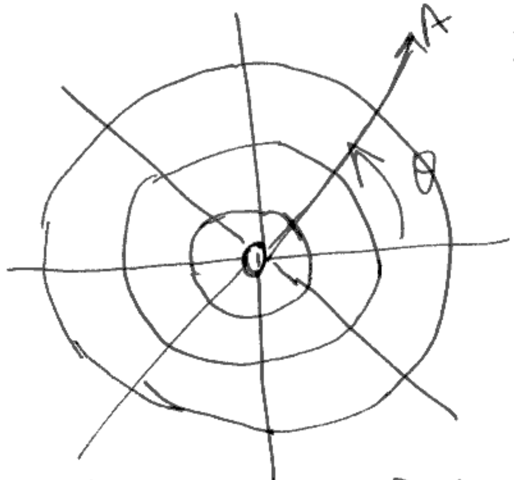
In contrast to this, plane polar unit vectors, though constant in magnitude (unit magnitude) vary in direction from point to point. This has the consequence that the time derivative of vectors in polar coordinates must also appropriately factor in non-vanishing time derivative of unit vectors $\hat{r}$ and $\hat{\theta}$. Thus, the expression for velocity and accelaration which were very straight forward in cartesian coordinates, now look very complicated in polar coordinates.

Q Why would one abandon the simplicity of cartesian coordinates in favour of more fancy but complicated polar coordinates?

ANSWER: One should abandon neither. It is rather a matter of choosing horses for courses. As Kleppner and Kolenkov put it - it is not that polar coordinates are more complicated but cartesian coordinates are simpler than they have the right to be - atleast for certain situations.



DOWNTOWN CHICAGO



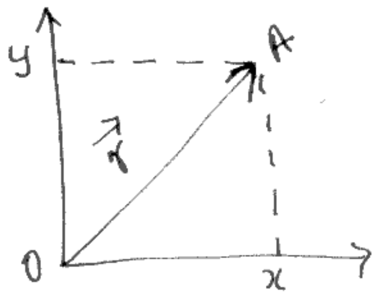
CONNOUGHT PLACE NEW DELHI

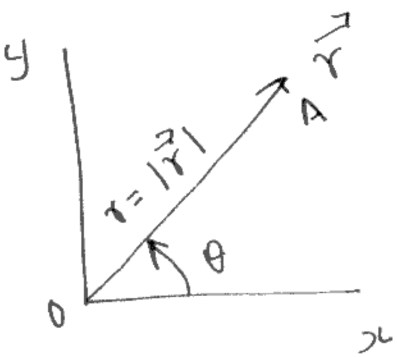
Figure on the left shows the street layout for downtown Chicago and Connought place, New Delhi. It is not difficult to imagine that polar coordinates are more suitable to describe the geometry of CP, New Delhi. If a car is going along the segment OA, the displacement is completely described as pure scaling of position vector without any change in angle θ. When viewed in cartesian system both x and y coordinates are changing. However, not both of them are independent as they are related by

$$\frac{y}{x} = \tan\theta = \text{constant}.$$

Such a constraint is already built into polar coordinate and hence it has only one free variable - r, distance from origin.

EXPRESSION FOR DISPLACEMENT VELOCITY AND ACCELARATION IN CARTESIAN & POLAR COORDINATES





Note: → Here the position vector $\vec{r}$ represents a generic vector. Like any vector, $\vec{r}$ has an existence INDEPENDENT of any coordinate system. Letter r in vector $\vec{r}$ has no affiliation whatsoever to any coordinate system.

→ Cartesian components of a vector $\vec{r}\,(= x\hat{i} + y\hat{j})$ are obtained by taking projections of $\vec{r}$ on x and y axis.

→ polar coordinate r is defined as length of vector $\vec{r}$, i.e. $r = |\vec{r}|$.

→ polar coordinate θ is defined as the angle between $\vec{r}$ and x-axis. θ is measured +ve, away and in counterclockwise direction from x-axis.

CARTESIAN SYSTEM

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \underbrace{\dot{x}}_{V_x}\hat{i} + \underbrace{\dot{y}}_{V_y}\hat{j} + \underbrace{x\dot{\hat{i}} + y\dot{\hat{j}}}_{=0}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \underbrace{\ddot{x}}_{a_x}\hat{i} + \underbrace{\ddot{y}}_{a_y}\hat{j}$$

$\hat{i}$ and $\hat{j}$ being truly constant unit vectors, their differentiation yields zero, resulting in very simple expression for $\vec{v}$ and $\vec{a}$.

POLAR COORDINATES

Going by the analogy suggested by cartesian system, we might resolve a vector in polar coordinates as follows

$$\vec{r} = r\hat{r} + \theta\hat{\theta}$$

THIS IS HOWEVER WRONG! θ being dimensionless, the dimensions on two sides do not match. Note that, unit vectors being ratios of vectors and their magnitudes, are by definition, dimensionless in every coordinate system.

THE CORRECT REPRESENTATION IS

$$\vec{r} = r\hat{r}$$

A vector afterall is uniquely specified by its magnitude and direction. Here $r = |\vec{r}|$ specifies its magnitude and $\hat{r}$ specifies direction of $\vec{r}$. If you are wondering where is θ dependence, then you must remember that the direction $\hat{r}$ is indeed a function of θ. That is

$$\vec{r} = r\hat{r}(\theta).$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\underbrace{\dot{\hat{r}}}_{\neq 0.}$$

since $\hat{r}$ is constant but points in different directions for different values of θ, $\dot{\hat{r}} \neq 0$. Let us find $\dot{\hat{r}}$ by two methods.

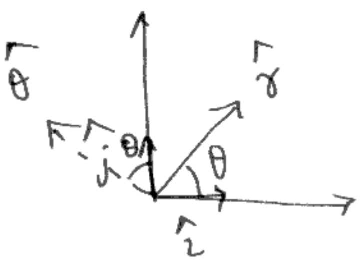
METHOD - 1.

Since $\hat{r}$ is of unit magnitude, the only way it can change in time is by pure rotation without scaling. We have already found the time derivative of such a vector and now we can borrow the result

$$\dot{\hat{r}} = |\hat{r}|\frac{d\theta}{dt}\hat{\theta}.$$ Thus,

$$\vec{V} = \dot{r}\hat{r} + r\underbrace{|\hat{r}|}_{1}\dot{\theta}\hat{\theta} = \underbrace{\dot{r}}_{V_r}\hat{r} + \underbrace{r\dot{\theta}}_{V_\theta}\hat{\theta}.$$

METHOD - 2



$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\dot{\hat{r}} = \dot{\theta}\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}}$$ Thus

$$\vec{V} = \underbrace{\dot{r}}_{V_r}\,\hat{r} + \underbrace{r\dot{\theta}}_{V_\theta}\,\hat{\theta}$$

Now,

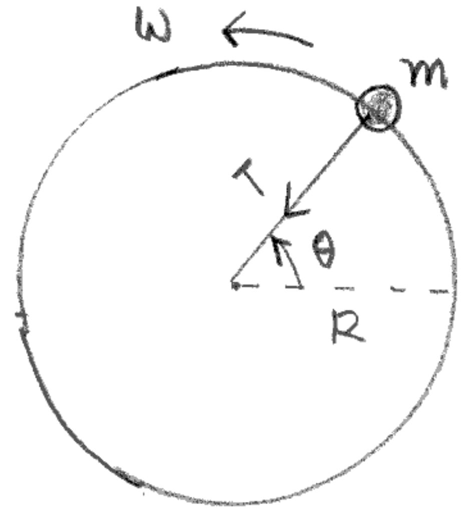
$$\vec{a} = \frac{d\vec{V}}{dt} = \ddot{r}\,\hat{r} + \dot{r}\,\dot{\hat{r}} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\dot{\hat{\theta}}$$

We need to find $\dot{\hat{\theta}}$ and then substitute for $\dot{\hat{r}}$, $\dot{\hat{\theta}}$, and then collect the coefficients of $\hat{r}$ and $\hat{\theta}$ to identify a_r and a_θ. Following method 1 or 2 above, we find

$$\boxed{\dot{\hat{\theta}} = -\dot{\theta}\,\hat{r}}$$ Thus,

$$\vec{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\,\hat{r} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\,\hat{\theta}.$$

Example 2.5 Block on a string in absence of gravity.



Block of mass m is tied to a massless inextensible string of and rotated in a circular path of radius R. What is the tension T?

Sol. We all know the answer: $T = \frac{mv^2}{R}$ and is directed radially inwards, $v = R\omega$. Let us understand it in the context of polar coordinates:

1) Circular geometry ⇒ polar coordinates are more suitable.
2) Identify all the forces and resolve them in radial and tangential.
3) Component in radially outward (inward) is positive (negative). Similarly counterclockwise (clockwise) is +ve (-ve).

3) Do not touch the signs of accelarations.

Thus, $F_r \hat{r} = m a_r \hat{r}$

$F_\theta \hat{\theta} = m a_\theta \hat{\theta}$.

radial| $\underbrace{-T}_{\substack{\text{radially}\\ \text{inwards}\\ \text{hence -ve}}} \hat{r} = m \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r} \hat{r}$

do not touch any sign here

Since $r = R \Rightarrow \dot{r} = 0,\ \ddot{r} = 0,\ \dot{\theta} = \omega$

$$\boxed{T = mR\omega^2 = \frac{mv^2}{R}} \quad \because v = R\omega$$

tangential| $\underbrace{0}_{\substack{\text{No force}\\ \text{in } \hat{\theta}}} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

$$\Rightarrow R\ddot{\theta} = -2\underset{\hookrightarrow 0}{\dot{r}}\dot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{d\omega}{dt} = 0 \Rightarrow \omega = \text{const}.$$

Remarks:

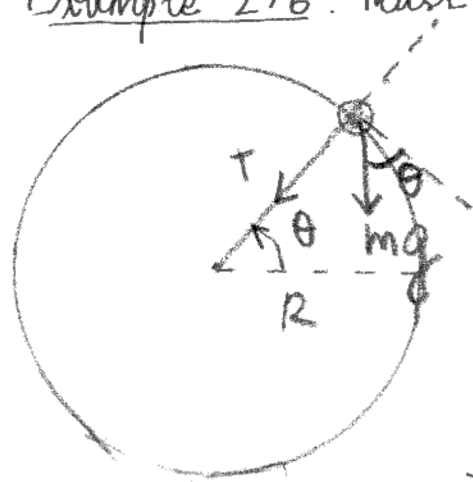
→ How did we use $v = R\omega$?
General expression for $\vec{v}$ in polar coordinates

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = 0$ here $|\vec{v}| = R\dot{\theta} = R\omega$.

→ In your class XII when you used centripetal accelaration to be v^2/R you were implicitly using polar coordinates.

Example 2.6. Mass on a string under gravity.



$\hat{r}|\ (-T - mg\sin\theta)\hat{r} = m(\underset{\hookrightarrow 0}{\ddot{r}} - r\dot{\theta}^2)\hat{r}$

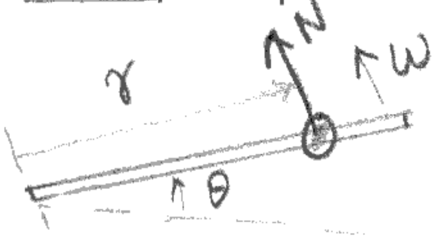
$\hat{\theta}|\ -mg\cos\theta\,\hat{\theta} = m(r\ddot{\theta} + 2\underset{\hookrightarrow 0}{\dot{r}}\dot{\theta})\hat{\theta}$

Please note the signs of the components of forces along radial and tangential directions.

$T = mR\omega^2 - mg\sin\theta$

Since T can never be -ve, $mR\omega^2$ must always be greater than mg (maximum value of $\sin\theta = 1$). When this condition fails $\dot{r} \neq 0$

2.33 A particle of mass m is free to slide on a frictionless thin rod. The rod rotates in a plane about one end at constant angular speed ω. Show that the motion is given by $r = Ae^{-\beta t} + Be^{+\beta t}$, where β is a constant which you must find, and A and B are arbitrary constants. Neglect gravity. Show that for a particular choice of initial conditions (that is $r(t=0)$ and $\dot{r}(t=0)$), it is possible to obtain a solution such that r decreases continually in time, but that for any other choice r will eventually increase.



Solution: Note: This problem as well as the next problem beautifully illustrate some of the peculiarities of polar coordinates

→ Since the rod is frictionless, there is no force ~~on~~ at all in the radial direction. The particle is ofcourse subject to normal reaction due to rod but it is in tangential direction and has no component in the radial direction.

MYSTERY: Despite the absence of radial force, the centripetal accelaration is non-zero.

$\hat{r}$] $\quad 0 = m(\underbrace{\ddot{r} - r\dot{\theta}^2}_{a_r}) \Rightarrow a_r = 0.$

$\hat{\theta}$] $\quad N\hat{\theta} = m(r\underbrace{\ddot{\theta}}_{\to 0} + 2\dot{r}\dot{\theta})$

Let us solve radial equation to find $r(t)$.

$$\ddot{r} = r\dot{\theta}^2 \quad \text{here } \dot{\theta} = \omega = \text{const}$$
$$\text{Thus } \ddot{\theta} = 0$$

Hence $r(t)$ is that function whose second derivative is constant times itself. What we have is a second order ordinary differential equation whose general solution will have two arbitrary constants, to be determined by specific initial conditions.

$r = e^{\beta t}$ fits the bill. Since it must satisfy our d.e.,

$$\dot{r} = \beta e^{\beta t}$$

$$\ddot{r} = \beta^2 \underbrace{e^{\beta t}}_{r(t)} = \omega^2 r(t)$$

$$\Rightarrow \quad \beta = \pm \omega .$$

Thus, for a 2^{nd} order linear d.e. there are two linearly independent solutions. A general solution is linear superposition of two. Thus

$$r(t) = A e^{-\omega t} + B e^{\omega t}.$$

~~To~~ To determine A and B, we need two equations which are obtained by specific initial conditions stating position and velocity at $t = 0$.

Say $r(t=0) = r_0$ $v(t=0) = v_0$. Thus,

$$r(t) = A e^{-\omega t} + B e^{\omega t} \Rightarrow r_0 = A + B.$$

$$\dot{r}(t) = -A\omega e^{-\omega t} + B\omega e^{\omega t} \Rightarrow v_0 = \omega(B - A).$$

Thus

$$\omega r_0 = (A+B)\omega$$

$$v_0 = (-A+B)\omega.$$

Adding, we get $B = \dfrac{\omega r_0 + v_0}{2\omega}$

Subtracting, we get $A = \dfrac{\omega r_0 - v_0}{2\omega}$.

Thus,

$$r(t) = \left(\frac{\omega r_0 - v_0}{2\omega}\right) e^{-\omega t} + \left(\frac{\omega r_0 + v_0}{2\omega}\right) e^{\omega t}.$$

For r to constantly decrease in time, dr/dt should be -ve.

$$\frac{dr}{dt} = -\frac{(\omega r_0 - v_0)}{2} e^{-\omega t} + \frac{(\omega r_0 + v_0)}{2} e^{\omega t}$$

Thus for $\frac{dr}{dt} < 0$, $(\omega r_0 + v_0) < 0$.

THE MYSTERIOUS PART OF 2.33

The mystery stems from the expectation that since $F_r = m a_r$ and since $F_r = 0$, a_r should be zero and hence there should be no dynamics in radial direction. We are shocked that despite $F_r = 0$, both the radial terms, $\ddot{r} \neq 0$ and $r\dot{\theta}^2 \neq 0$.

<u>RESOLUTION</u>: Problem lies with our intuition that borrows heavily from our cartesian coordinate experience. There, ~~if~~ $F_x = 0 \Rightarrow a_x = 0$ $\Rightarrow \ddot{x} = 0$, because $a_x = \ddot{x}$.

In polar coordinates, $a_r \neq \ddot{r}$, rather

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

Thus, $F_r = m a_r$, sure $\Rightarrow a_r = 0$, but a_r can be zero in two ways

1) $\ddot{r} = 0, r\dot{\theta}^2 \Rightarrow$ Trivial ~~dy~~ case as no dynamics

2) $\ddot{r} = r\dot{\theta}^2 \Rightarrow$ Non-trivial dynamics

so let us try to understand how $\ddot{r} \neq 0$ and $r\dot{\theta}^2 \neq 0$ despite $F_r = 0$.

My contention is that only way to logically accomodate the observed fact that $\dot{\theta} \neq 0$ and $F_\theta = N \neq 0$ is to have $\dot{r} \neq 0$, $r\dot{\theta}^2 \neq 0$ and $\ddot{r} \neq 0$.

→ Suppose at $t = 0$, the position vector of ~~pos~~ the particle is

$$\vec{r} = r\hat{r}$$

Some undeniable observations

1) The mass is compelled to rotate with the rod hence
   - → $\dot{\theta} \neq 0$
   - → $\hat{r}$ is changing direction thus $\dot{\hat{r}} = \dot{\theta}\hat{\theta} \neq 0$
   - → The only force on the mass is normal reaction: $F_\theta = N$.

Now $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$.

Naively, since $F_r = 0$ we do not expect any motion in radial direction so let us ~~take~~ put $\dot{r} = 0$, and where does it lead us.

so $\vec{v} = r\dot{\theta}\hat{\theta}$

$$\vec{a} = \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$= \underbrace{-r\dot{\theta}^2}_{a_r}\hat{\theta} + \underbrace{(\underbrace{\dot{r}\cancel{\dot{\theta}}}_{\to 0 \text{ (we assumed)}} + \underbrace{r\ddot{\theta}}_{\to 0 \ (\because\ \dot{\theta} = \omega = \text{const.})})}_{a_\theta}\hat{\theta}$$

Thus, we ~~we~~ have $a_r = -r\dot{\theta}^2 \neq 0$ and ~~$a_\theta = \dot{r}\dot{\theta} \neq$~~ $a_\theta = 0$.

Now $F_r = m a_r$ hence $F_r \neq 0$.

But there is no physical agency (since $\mu = 0$ and $g = 0$) that can provide $F_r \neq 0$. Hence, we reach a farcical conclusion that

$$0 = -mr\dot{\theta}^2.$$

The only way to save this is to assume that the other term in radial accelaration $\ddot{r} \neq 0$, so that we can have

$$0 = \ddot{r} - r\dot{\theta}^2 = 0$$

. Makes sense!

However, just a while ago that $\dot{r} = 0$ when wrote down $\vec{v}$. Now we are forced to admit that $\dot{r} \neq 0$ since $\ddot{r} \neq 0$.

Thus, the correct expression for $\vec{v}$ is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

But there may still be skeptics amongst you who are not willing to accept $\dot{r} \neq 0$ since $F_r = 0$. Here is another arguement for them. We just now found that

$$a_\theta = \underbrace{\dot{r}}_{\text{You are forcing to be zero}}\dot{\theta} + \cancel{2r}\,\underbrace{r\ddot{\theta}}_{\to 0\ \omega = \text{const}}$$

Thus $a_\theta = 0 \Rightarrow F_\theta = m a_\theta = 0$.

But we agreed that there is non-zero normal reaction $N = F_\theta$ providing tangential force and compelling the particle to move with the rod. The only term in a_θ, that can provide for $N \neq 0$ is $\dot{r}\dot{\theta}$ term as $\ddot{\theta} = 0$. Thus $\dot{r} \neq 0$.

Hence $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\bar{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\to 0}\hat{r} + (\underbrace{2\dot{r}\dot{\theta}}_{N/m} + \underbrace{r\ddot{\theta}}_{\to 0})\hat{\theta}.$$

ANGULAR MOMENTUM PERSPECTIVES ON 2.33.

→ Mass m is at a distance r from the pivot of the rod and has non-vanishing tangential force N. Thus, it is acted upon by a torque

$$\tau = r(t)N = \frac{dL}{dt} \qquad L = \text{Angular momentum}, \quad L = I\omega.$$

Naively, we would tend to equate

$$\tau = I\ddot{\theta} \qquad I = mr^2.$$

but $\dot{\theta} = \text{constant} \Rightarrow \ddot{\theta} = 0 \Rightarrow \tau = 0.$

But $L = I\omega$

$$\tau = \frac{dL}{dt} = \dot{I}\omega + \underbrace{\dot{\omega}}_{\to 0} I$$

$$\tau = \frac{d}{dt}(mr^2)\,\omega$$

$$\tau = 2mr\dot{r}\omega = rN$$

~~But~~ $\Rightarrow N = 2m\dot{r}\omega.$

Now you understand that naively putting $\dot{r} = 0 \Rightarrow N = 0 \Rightarrow \tau = 0 \Rightarrow L = \text{const} \Rightarrow r = \text{const} \Rightarrow r\dot{\theta}^2$ is nonzero despite $F_r = 0 = \ddot{r}$.

UPSHOT is once you have a rotating $\hat{r}$, $\Rightarrow \dot{\hat{r}} \propto \hat{\theta}$ and $\dot{\hat{\theta}} \propto \hat{r}$. Thus the time derivative of unit vectors feed the dynamics from $\hat{\theta}$ ($N\hat{\theta} \neq 0$) into $\hat{r}$ ($\ddot{r} \neq 0, r\dot{\theta}^2 \neq 0$). Can we have a situation in which real forces in ~~dy~~ $\hat{r}$ direction feed the dynamics into $\hat{\theta}$ direction? Yes, this is precisely the situation in problem 2.34, That is our next problem. But before that

2.33 FROM ~~ROTA~~ THE PERSPECTIVES OF NON INERTIAL FRAME.

In the frame of someone rotating with the rod, the particle is not rotating at all. However there is pseudo-force $mr\dot{\theta}^2$ in radially outward direction. Thus, equation of motion is

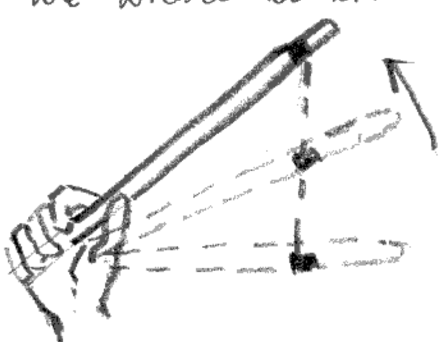
$$mr\dot{\theta}^2 = ma = m\ddot{r} \Rightarrow \ddot{r} = r\dot{\theta}^2$$

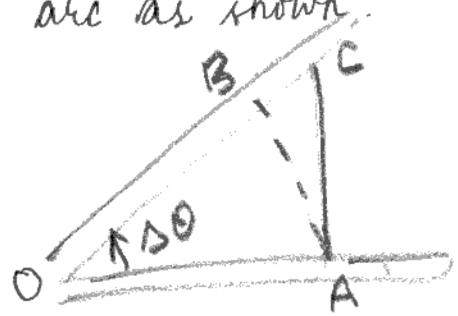
Isn't this what we got from inertial frame perspective.

2.33 : A PHYSICAL PERSPECTIVE

aka, How to reset a mercury thermomter or drain clean a garden hose :

The mystefying crux of 2.33 was radial motion without radial motion. We confront a similar physical situation when we want to get rid of water from the garden hose or wants the mercury to reset to normal level after use. How do we do it? Not by shaking the tube longitudinally but by whirling the tube in a circular arc. Consider a water droplet on the frictionless inside wall of a standard garden hose. To expel it we whirl it in a arc as shown.





The water drop which was initially at A, under the influence of the normal force of the movement of hose, travels tangentially and reaches point C. Now, from the perspective of an inertial observer, it has rotated by an angle $\Delta\theta$, as well as gone radially outward by a distance $\Delta r = BC$. As viewed from the rotating frame of hose, it has only moved radially outward by $\Delta r = BC$.

Question: What is accelaration of water in rotating frame as it goes distance BC.

$BC = OC - OB$. $(OA = OB = r)$.

OAC being a right triangle

$BC = r\sec\Delta\theta - r$.

Now $\sec\Delta\theta = (\cos\Delta\theta)^{-1} \approx [1 - \frac{1}{2}(\Delta\theta)^2]^{-1}$

$\approx 1 + \frac{1}{2}(\Delta\theta)^2$

$\Rightarrow BC \approx \frac{1}{2} r (\Delta\theta)^2$. $\quad \Delta\theta = \omega\Delta t$

$BC \doteq \frac{1}{2} r \omega^2 (\Delta t)^2$.

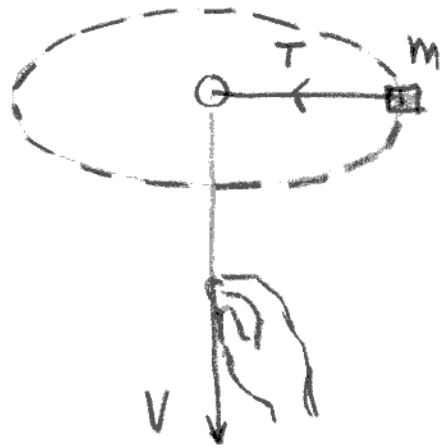
Comparing with $s = \frac{1}{2}at^2 \Rightarrow a = r\omega^2$.

Thus in rotating frame $a = \ddot{r} = r\omega^2$.

In an inertial frame: $\ddot{r} - r\omega^2 = 0$ as expected.

Lesson: If you strip a phenomena of superficial differences, underlying physics may be same.

2.34 Mass m whirls around on a string which passes through a ring as shown. Neglect gravity.



Initially the mass is at a distance r_0 and is revolving at a angular velocity ω_0. The string is pulled at a constant velocity V starting at $t=0$ so that the radial distance to the mass decreases. Draw a force diagram and obtain a differential equation for ω. Find a) $\omega(t)$, b) The force needed to pull the string.

sol: $\hat{r}$ $\quad -T\hat{r} = m(\ddot{r} - r\dot{\theta}^2)\hat{r}$

Since the string is being pulled with constant speed V,

$$\frac{dr}{dt} = -V \Rightarrow r = r_0 - Vt$$

$$\ddot{r} = 0.$$

Thus, $T = m\,r(t)\,\omega^2$

The force needed to pull the string is this tension T. Is T increasing or decreasing with time? Since r is decreasing it seems to suggest that T is decreasing. But since we are pulling T must be increasing with time. This means, not only is $\omega(t)$ a function of t, but it is sufficiently increasing function of time, to ensure that T is rising despite decrease in time.

MYSTERY: It is clear that there is no force in tangential direction we ~~no~~'d expect tangential acceleration to be zero. But we just concluded that $\dot{\omega} = \ddot{\theta} > 0$. How come we succeed in giving tangential acceleration by pulling radially inwards? This problem is an anti-thesis of 2.33, where $\ddot{r} \neq 0$ despite $F_r = 0$. Here $\ddot{\theta} \neq 0$, $F_\theta = 0$.

The resolution of mystery is also similar.

$F_\theta = 0 \Longrightarrow a_\theta = 0.$

But $a_\theta = 0$ in two ways

1) Both $r\ddot{\theta} = 0$ and $\dot{r}\dot{\theta} = 0$
This is a trivial case with no dynamics

2) $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ Non-trivial
$\Rightarrow r\ddot{\theta} = -2\dot{r}\dot{\theta}$ dynamics

The dynamics of radial equation already suggested the need for differential equation for ω. This is provided by tangential equation. Thus,

$$F_\theta = 0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

or $$\frac{r\,d\omega}{dt} = -\frac{2\dot{r}\omega}{r} \qquad \left[\frac{dr}{dt} = -V\right]$$

$$\int_{\omega(r_0)}^{\omega(r)} \frac{d\omega}{\omega} = -2\int_{r_0} \frac{dr'}{r'} \qquad \left[dt = -\frac{dr}{V}\right.$$

$$\ln\left[\frac{\omega(r)}{\omega(r_0)}\right] = -2\ln\left[\frac{r(t)}{r_0}\right]$$

$$\omega(r) = \frac{\omega(r_0)\,r^2(t)}{r_0^2}$$

$$\boxed{\omega(t) = \frac{\omega(r_0)\,(r_0 - Vt)^2}{r_0^2}}$$

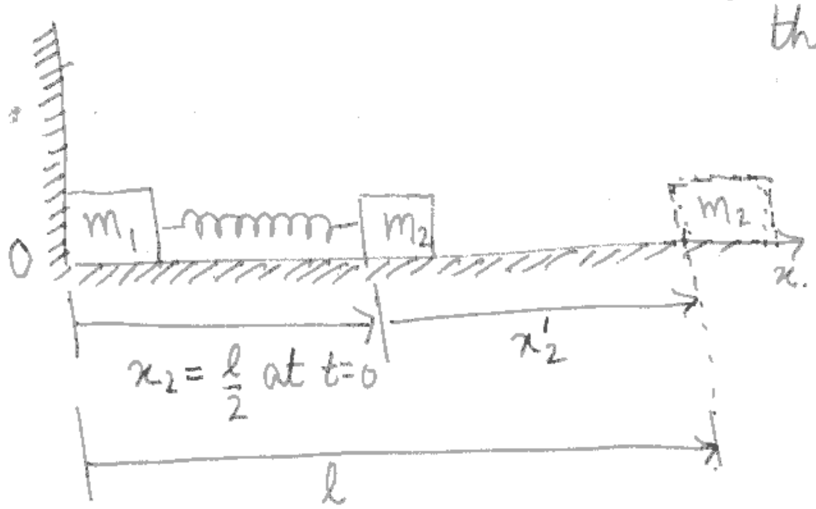
$$T(t) = m\,r(t)\,\omega^2(t)$$
$$= \frac{m\,\omega^2(r_0)\,(r_0 - Vt)^5}{r_0^4}$$

Note: The same result could have been obtained in two steps using principle of conservation of angular momentum (since the only force T passes through origin, has no moment arm, and hence cannot exert torque). Conservation laws, though powerful, hide a lot of details which are revealed by dynamical method. Both are crucial for complete understanding.

Problem 3.7 (Notes: Rishikesh Vaidya).
(K & K).

A system is composed of two blocks of masses m_1 & m_2, connected by a massless spring with spring constant k. The blocks slides on a frictionless plane. The unstretched length of the spring is l. Initially m_2 is held so that the spring is compressed to $l/2$ and m_1 is forced against a stop. m_2 is released at $t=0$. Find the motion of the center of mass of the system as a function of time.

Solution: When measured from the origin O at the wall.



At $t = 0$

$x_1 = 0$

$x_2 = l/2$.

Until m_2 reaches $x_2 = l$ it moves solely under the influence of spring force, with frequency $\omega = \sqrt{k/m_2}$, $T = 2\pi/\omega$.

Hence, for $0 < t < T/4$.

m_1

$\xleftarrow{F_s}\xrightarrow{N}$

For m_1 $\quad N - F_s = 0$

so $x_1 = 0$.

For m_2, the equation of motion is

$$m_2\ddot{x}_2 = k\underbrace{(l-x_2)}_{\text{compression of spring.}}$$

} +ve sign in $+k(l-x_2)$ is because the direction of spring force is same as that of increase in x_2.

Lets say $l-x_2 = x_2' \Rightarrow \ddot{x}_2 = -\ddot{x}_2'$

Thus $m_2\ddot{x}_2' = -kx_2'$

This is an equation for a simple harmonic motion. It is a 2nd order (in derivative) linear differential equation. It is linear because every term contains first (linear) power in x. It has the property (due to linearity) that any linear combination of solution is also a solution.

Thus the most general solution is

$$x_2' = A\cos\omega t + B\sin\omega t$$

} ∵ sin & cosine both satisfy the linear d.e. above.

A and B are arbitrary constants fixed by initial conditions (values of x_2' and $\dot{x}_2'$ at $t=0$).

$$x_2'(t=0) = \frac{l}{2} = A\cos(\omega.0) + B\sin(\omega.0) = A + 0.$$

$$\Rightarrow \boxed{A = l/2}$$

$$\dot{x}_2'(t=0) = 0 = -A\omega\sin(\omega.0) + B\omega\cos(\omega.0)$$

$$\Rightarrow \boxed{B=0}$$

Thus $\boxed{x_2' = \frac{l}{2}\cos\omega t}$ & $\boxed{\begin{aligned} x_2 &= l - x_2' \\ &= l\left(1-\tfrac{1}{2}\cos\omega t\right)\end{aligned}}$

Thus for $0<t<T/4$

$$R_{CM} = \frac{m_2 x_2}{m_1+m_2} = \frac{m_2 l}{m_1+m_2}\left(1-\frac{1}{2}\cos\omega t\right).$$

$$\dot{R}_{CM} = \frac{m_2 \dot{x}_2}{m_1+m_2} = \frac{m_2 l \omega}{2(m_1+m_2)}\sin\omega t.$$

Now for $t \geq T/4$.

At $t = T/4$, the spring is at its natural length and hence mass m_1 is free from both, the spring force as well as normal reaction of wall.

For $t \geq T/4$, there are no external forces on the system of masses and the center of mass moves with a constant speed it had $t = T/4$.

$$\dot{R}_{CM}\Big|_{t=T/4} = \frac{m_2 l\omega}{2(m_1+m_2)}\sin\omega\cdot\frac{2\pi}{\omega\cdot 4} = \frac{m_2 l\omega}{2(m_1+m_2)}$$

$$R_{CM}(t>T/4) = R_{CM}\Big|_{t=T/4} + \dot{R}\Big|_{t=T/4}\times t.$$

$$= \frac{m_2 l}{m_1+m_2} + \frac{m_2 l\omega t}{2(m_1+m_2)}.$$

$$\boxed{R_{CM}(t>T/4) = \frac{m_2 l}{m_1+m_2}\left[1+\frac{\omega t}{2}\right]}$$

LECTURE - 6

→ MASS VARYING SYSTEMS.

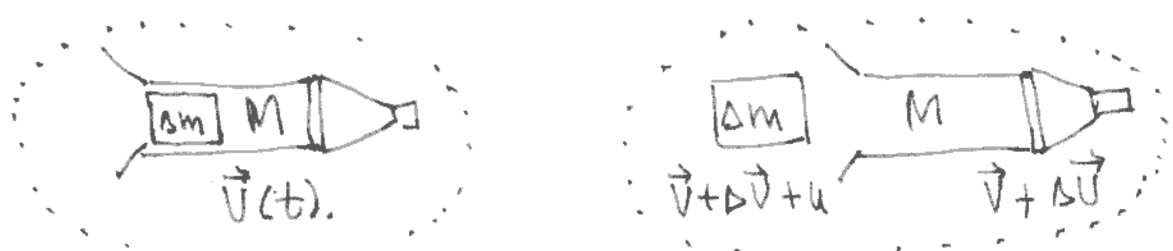
Newton's second law: $\vec{F} = m\vec{a}$.

When m changes with time, this form poses problem.

Better use $\vec{F} = \frac{d\vec{P}}{dt}$. How?

- Compare the momentum at two different times (infinitesimally apart), ensuring that you keep same mass at ~~the~~ both the instants, thus also taking account of momentum flowing into or out of the system.
- All measurements of velocity must be referred to inertial frames.

Example: Rocket equation



↑System at t.
Mass: M+ΔM.

↑System at t+Δt.
Mass: M+ΔM.

$\vec{u}$: velocity of exhaust w.r.t. rocket

$\vec{V}+\Delta\vec{V}+\vec{u}$: velocity of exhaust w.r.t. inertial frame.

Note the vector addition of velocities in $\vec{V}+\Delta\vec{V}+\vec{u}$.

Comparing momentum at t and $t+\Delta t$.

$$\vec{P}(t) = (M+\Delta m)\vec{V}(t)$$

$$\vec{P}(t+\Delta t) = M(\vec{V}+\Delta\vec{V}) + \Delta m(\vec{V}+\Delta\vec{V}+\vec{u}).$$

$$\Delta\vec{P} = \vec{P}(t+\Delta t) - \vec{P}(t).$$

$$= \cancel{M\vec{V}(t)} + M\Delta\vec{V} + \cancel{\vec{V}\Delta m} + \vec{u}\,\Delta m - \cancel{M\vec{V}(t)} - \cancel{\vec{V}(t)\Delta m}$$

(Here we have neglected the product of second order differentials such as $\Delta m \Delta \vec{V}$ which will vanish when we take $\Delta t \to 0$).

$$\Delta\vec{P} = M\Delta\vec{V} + \vec{u}\,\Delta m = \vec{F}_{ext}\Delta t.$$

Thus $$\vec{F}_{ext} = \lim_{\Delta t\to 0}\frac{\Delta\vec{P}}{\Delta t} = M\frac{d\vec{V}}{dt} + \vec{u}\frac{dm}{dt}.$$

Remarks: 1) Here M denotes the total mass of the rocket + exhaust at any given time t. So $M \equiv M(t)$.

2) $\frac{dm}{dt} \equiv$ rate at which the rocket is losing the mass. Thus expressed in terms of the total instantaneous mass of the rocket, we have $\boxed{\frac{dM}{dt} = -\frac{dm}{dt}}$

3) Note that $\vec{u}$ denotes velocity of exhaust relative to the rocket. May be we can write it as: $\vec{u} = \vec{V}_{rel}$. Thus

$$\boxed{F_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}}$$

This is Newton's 2nd law applied to any mass varying system.

Example: Rocket moving in a free space.

$$F_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}.$$

$F_{ext} = 0$ (free space) $\Rightarrow$

$$M(t)\frac{d\vec{V}}{dt} = \vec{V}_{rel}\frac{dM(t)}{dt}.$$

$$\Rightarrow \int_{V_0}^{V_f} dV = \vec{V}_{rel}\int_{M_0}^{M_f}\frac{dM}{M}$$

$$\vec{V}_f - \vec{V}_0 = \vec{V}_{rel}\ln\left(\frac{M_f}{M_0}\right).$$

If $V_0 = 0$ and $\vec{V}_{rel}$ is always opposite to $\vec{V}$.

$$\boxed{V_f = V_{rel}\ln\left(\frac{M_0}{M_f}\right)}$$

The final velocity is thus independent of how the mass is released - rapidly or slowly. It only depends on the exhaust speed and the ratio of initial to final mass.

Rocket in constant gravitational speed ($\vec{F}_{ext} = M\vec{g}$).

$$\vec{F}_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM}{dt}$$

$$-\cancel{M}g = \cancel{M}\frac{dV}{dt} + \frac{V_{rel}}{M}\frac{dM}{dt}.$$

$$\int_{V_0}^{V_f} dV = -V_{rel}\int_{M_0}^{M_f}\frac{dM}{M} - g\int_{t_0}^{t_f} dt$$

$$V_f = V_{rel}\ln\left(\frac{M_0}{M_f}\right) - g(t_f).$$

Thus faster you burn your fuel higher is your V_f

$\boxed{3.20}$ A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u. Assume that the rate at which fuel is expelled is $dM/dt = -|r|M$, where M is the instantaneous mass of rocket and $|r|$ is a constant, and that the rocket is retarded by air resistance with a force Mbv, where b is a constant. Find the velocity of rocket as a function of time.

sol.
$$\vec{F}_{ext} = M(t)\frac{dV(t)}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}.$$

$$-\cancel{M(t)}g - b\cancel{M}v = \cancel{M}\frac{dv}{dt} - (+u)\cdot(+)|r|\cancel{M}.$$

$$-g - bv = \frac{dv}{dt} - ru.$$

$$\frac{dv}{dt} = \underbrace{\underbrace{ru - g}_{\alpha} - bv}_{v'} \qquad \alpha = ru - g$$

$v' = \alpha - bv$

$dv' = -b\,dv.$

$v: 0 \to v_f$

$v': \alpha \to \alpha - bv_f.$

$$-\frac{1}{b}\int_{\alpha}^{\alpha - bv_f}\frac{dv'}{v'} = \int dt.$$

$$\ln v'\Big|_{\alpha}^{\alpha-bv_f} = -bt.$$

$$\frac{\alpha - bv_f}{\alpha} = e^{-bt}.$$

$$\frac{\alpha}{b}[1 - e^{-bt}] = v_f.$$

$$\Rightarrow \boxed{\frac{ru-g}{b}\left[1 - e^{-bt}\right] = v_f}.$$

Ch. 4. WORK AND ENERGY (Lecture 7).

Carrying forward from the initial remarks in slides.

$\vec{F} = m\vec{a}$ can in principle solve everything. In practice however we are stuck because

$$\vec{F}(\vec{r}) = \frac{d\vec{V}(t)}{dt}$$

To integrate this with respect to time we need to know force as a function of time whereas very often we know it as a function of position. If we somehow manage to solve this integral, we automatically land into the notion of energy (as we will soon see) as the first integral of motion. Energy in turn derives its usefulness from its conservation and conversion. The question then is - What is the connection between energy and force, since both are capable of giving information about physical system. To understand this question, we must pose another question —

What does a force $\vec{F}$ do?

Example: We will analyze this question in terms of the effect of a constant force F on a particle of mass m moving in 1-D.

$$F = \frac{dP}{dt} = \frac{m\,dV}{dt} = ma$$

Say, the force is acting for time t. Multiply both the sides by t.

$$Ft = mat = m(V_2 - V_1).$$

If the force F(t) is varying with time, then

$$F(t) = \frac{m\,dV}{dt}$$

$$\boxed{\int_{t_1}^{t_2} F(t)\,dt = m\int_{V_1}^{V_2} dV = m(V_2 - V_1).} \qquad (1)$$

Thus, the effect of force is in terms of its impulse (or its time integral) which results in the change in momentum. Since the change in momentum can be measured, Impulse is a good measure of the effect of force.

Now, suppose the particle covers a distance x, when the force acts on it (assume constant F). Multiplying both the sides of F = ma by x, now

$$Fx = max$$

$$= ma\left(\frac{V_1 + V_2}{2}\right)t$$

$$= \frac{1}{2}m(V_2 - V_1)(V_2 + V_1)$$

$$\boxed{Fx = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2} \qquad (2)$$

$\Rightarrow$ Work done by F $\equiv$ Change in K.E.

If F is not constant

$$\int_{x_1}^{x_2} F\,dx = m\int_{V_1}^{V_2} \frac{dV}{dt}\,dx$$

But $\frac{dV}{dt}dx = \frac{dV}{dx}dx \cdot \frac{dx}{dt} = V\,dV$

$$\boxed{\text{Thus, } \int_{x_1}^{x_2} F\,dx = m\int_{V_1}^{V_2} V\,dV = \frac{1}{2}m(V_2^2 - V_1^2)} \quad (2)$$

This is another way to characterize the impact of a ~~measure~~ force. Relation (1) & (2) are obtained by integrating F with respect to time and space and apriori, there is no reason to choose one over the other. However, the apparent similarity is deceptive for the following reasons.

1) Unlike t, $\vec{x}$, $\vec{P}$, and $\vec{F}$ are all vectors. This means that LHS of 1 (and hence RHS) will always be a vector.
2) Since $\vec{F}$ and $d\vec{r}$ are vectors, as you go to 2-D or 3-D, the effect of $\bar{F}$ on the mass will depend on the angle between $\vec{F}$ and $d\vec{r}$. For instance in circular motion F is constantly applied and it changes the momentum continuously but the magnitude of velocity remains unchanged.

(3) Given that $\vec{F}$ and $d\vec{r}$ are both vectors and there are two different ways to combine two vectors (dot and cross product) how do we generalize (2) as we go from 1-D to 2-D, 3-D? Ofcourse we know in hindsight that LHS is work done by $\vec{F}$ and hence we should have $\vec{F}.d\vec{r}$, but even without that, the RHS in 1-D case provides hint. Since it involves difference of two terms both quadratic in velocities it better be a scalar product (because $\vec{v}\times\vec{v}=0$). Since RHS is a scalar, LHS, better be a scalar, and $W = \int \vec{F}.d\vec{r}$.

(4) Here is another perspective on the space integral of F. We are trying to ask - what is the effect of F applying a force in an arbitrary direction, on the motion of an object? If apply F normal to the direction of motion we only change the direction of velocity without affecting its magnitude (as in circular motion). Applying force in a parallel direction changes only magnitude of velocity without affecting direction. This is precisely what RHS is about and hence we better use $F_{\parallel}$ i.e.. dot product $\vec{F}.d\vec{r}$.

Thus, starting from $\vec{F}(\vec{r}) = m\frac{d\vec{v}}{dt}$

we consider what happens when a particle moves a short distance $\Delta\vec{r}$ (during which $\vec{F}$ is effectively constant), and take a scalar product $\vec{F}.\Delta\vec{r}$. thus,

$$\vec{F}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\Delta\vec{r}$$

This step seems to assume that we know the entire trajectory and hence $\Delta\vec{r}$ before hand. Isn't this what we seek as our solution? Though an important objection, let us presume we know the trajectory and move ahead. For sufficiently short path, $\Delta\vec{r} = \vec{v}\,\Delta t$. thus,

$$m\frac{d\vec{v}}{dt}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\vec{v}\Delta t$$

[An aside on a vector identity. Let $\vec{A}$ and $\vec{B}$ be two vectors: then

$$\frac{d}{dt}[\vec{A}.\vec{B}] = \frac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$$

If $\vec{A} = \vec{B}$

$$\frac{d}{dt}[A^2] = 2\vec{A}.\frac{d\vec{A}}{dt}.$$

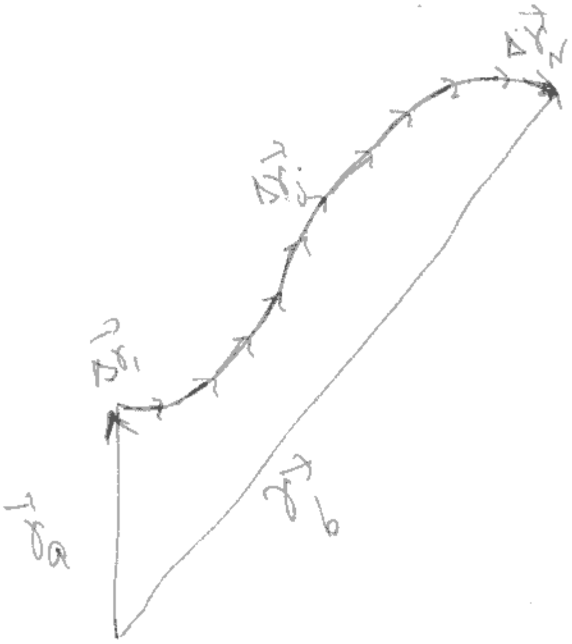
Remarks 1) If vector $\vec{A}$ stays constant in magnitude then LHS = 0. This implies $\frac{d\vec{A}}{dt}$ is $\perp$ $\vec{A}$. That is the only way $\vec{A}$ can change is rotate.

2) It assumes that dot product of two vectors is commutative]

Using $\frac{d}{dt}[A^2] = 2\vec{A}\cdot\frac{d\vec{A}}{dt}$

$$\vec{V}\cdot\frac{d\vec{V}}{dt} = \frac{1}{2}\frac{d}{dt}(v^2).$$

$$\vec{F}\cdot\Delta\vec{r} = \frac{m}{2}\frac{d}{dt}(v^2)\,\Delta t.$$



If we divide the entire trajectory from the initial position r_a to final position r_b into N short segments of length Δr_j. Then

$$\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \frac{m}{2}\frac{d}{dt}(v_j^2)\,\Delta t_j$$

Adding the equations of all segments

$$\sum_{j=1}^{N}\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \sum_{j=1}^{N}\frac{m}{2}\frac{d}{dt}(v_j^2)\,\Delta t_j$$

Taking the limit $\Delta r_j \to 0$, $N \to \infty$.

$$\int_{r_a}^{r_b}\vec{F}\cdot d\vec{r} = \int_{t_a}^{t_b}\frac{m}{2}\frac{d}{dt}(v^2)\,dt = \frac{m}{2}\int_{t_a}^{t_b}\frac{d}{dt}(v^2)\,dt$$

$$\boxed{\begin{aligned}\int_{r_a}^{r_b}\vec{F}\cdot d\vec{r} &= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2\\ W_{ba} &= K_b - K_a.\end{aligned}}$$

This is the famous Work-Energy theorem. We also ~~eta~~ established the connection between energy and Force. Actually this only half the connection.

LECTURE (9-10)

We have seen that there are two broad classes of forces

a) Conservative forces (e.g. all fundamental forces)

b) Non-conservative forces (e.g. friction, viscous drag etc).

Often both kinds of forces are at work - for instance an object falling through air experiences gravity as well as viscous drag. Moreover you may be wondering as to what does non-conservation mean? After all energy cannot vanish into thin air. And what about the Work-energy theorem?

Let us write the total force acting on a body as a sum of two parts.

$$\vec{F} = \vec{F}_c + \vec{F}_{NC}$$

{ C and NC obviously stand for conservative & non conservative forces.

The good part! Work energy theorem is true whether or not the force is conservative or not. The total work done by a force $\vec{F}$ as the particle moves from a to b is:

$$W_{ba}^{total} = \oint_a^b \vec{F}\cdot d\vec{r}$$

$$= \oint_a^b \vec{F}^c\cdot d\vec{r} + \oint_a^b \vec{F}^{NC}\cdot d\vec{r}$$

$$= -U_b + U_a + W_{ba}^{NC}.$$

{ Note $\oint$, here the curve on the integral stand for C for curve along which integral is carried out. A circle $\oint$ on the integral means the integral is around a closed path. BOTH ARE DIFFERENT!

Here U is the P.E associated with the conservative force. For the work done by non-conservative, one cannot associate a function of position such as potential energy. Since Work-energy theorem is always true, $W_{ba}^{total} = K_b - K_a$, thus

$$-U_b + U_a + W_{ba}^{NC} = K_b - K_a .$$

or

$$\boxed{K_b + U_b - (K_a + U_a) = W_{ba}^{NC}}$$

If we define <u>total mechanical energy</u> $E = K + U$, then E is no longer conserved, but depends on the state of the system.

$$\boxed{E_b - E_a = W_{ba}^{NC} .}$$

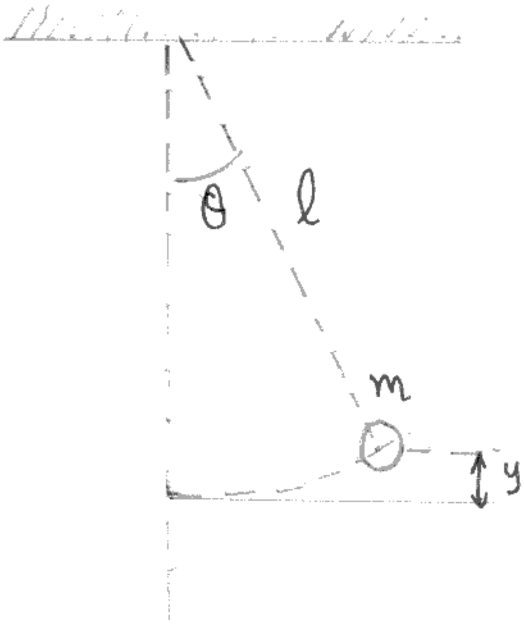
This is a generalization of the statement of conservation of total mechanical energy (word mechanical being important) to the case where non-conservative forces are present. Work done by non-conservative forces is dissipated as heat. As far as total mechanical energy is concerned it is Lost. However, if we take this loss into account, then total energy is always conserved (as it should be).

JUST HOW USEFUL IS ENERGY PERSPECTIVE

Newton's laws of motion and energy methods offer two different approaches to solve problems of dynamical systems. From the standpoint of mechanics, the two approaches are equivalent. However, as we discussed in the class, conservation laws follow directly from the symmetry properties of the transformations in space-time (details are beyond the scope of this course) and hence, in some sense more fundamental than Newton's laws which break down at high speeds (Well Newton's second still holds good, it is the Newton's conception of absolute and independent notion of space-time that is challenged) and atomic scales. In both these regimes conservation laws hold true.

ENERGY SOLUTION TO A DYNAMICAL PROBLEM ~~NOTE~~

In class we had solved the problem of simple harmonic motion of a mass spring system (problem 3.7) using Newton's 2nd law. To illustrate the power of energy method, we will now solve the problem of simple pendulum using energy methods. As you will learn, this method offers far more insight than you can glean from Newton's 2nd law.



The work done by gravitational force $-m\vec{g}$ on mass m, ~~is $m\vec{g}$~~ as it moves from $y=0$ to $y=y$ is $U(y)-U(0)=mgy$. The total energy of pendulum at any θ is

$$E = K + U$$
$$= \frac{1}{2} m l \dot{\theta}^2 + \cancel{\tfrac{1}{2}}\, mgy.$$

Here l = length of pendulum
$$y = l(1-\cos\theta).$$

At the end of the swing, say $\theta = \theta_0$ and $\dot{\theta} = 0$. Since there are no non-conservative forces, total energy is conserved. So

$$\frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0).$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \quad \text{———} \; ☆$$

which can rearranged to give

$$\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{2g}{l}} \int dt$$

Let us look at the solution for the case of small amplitude, so that, $\cos\theta \approx 1 - \theta^2/2$

$$\int \frac{d\theta}{\sqrt{\frac{1}{2}}\sqrt{\theta_0^2 - \theta^2}} = \sqrt{\frac{2g}{l}} \int dt.$$

8½

Introducing $\omega = \sqrt{g/l}$, we can rewrite

$$\int \frac{d\theta/\theta_0}{\sqrt{1-(\theta/\theta_0)^2}} = \omega \int dt \qquad \left\{ \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \right.$$

Put $x = \theta/\theta_0$

Taking the lower limits of integration $(\theta=0, t=0)$ and upper limits to be (θ, t).

$$\sin^{-1}\theta/\theta_0 - 0 = \sqrt{\frac{g}{l}}(t-0)$$

$$\boxed{\theta = \theta_0 \sin\omega t}$$

This is what we would obtain by solving $\bar{F} = m\bar{a}$. More importantly, the starred eq^n (*) on the previous page is a general equation that is not limited to the small-angle approximation It has mathematically exact solution in terms of functions called elliptic integrals. Even without going into that complexity we can use equation (*) to find an important result: that is, correction to the pendulum period due its finite amplitude. Such a correction would be very difficult to extract starting with the Newtonian equation of motion.

TWO IMPORTANT QUESTIONS

1) What distinct advantage energy method had to offer the bonus advertised above?

Ans: Did you notice that as opposed to solving Newton's second order differential equation, we integrated only once here. This is because when you start with energy equation, half of the problem is already solved for you, as you exploit work energy theorem. When you ~~w~~ further write work-done by a conservative force in terms of a potential energy function, you are exploiting energy as first ~~int~~ integral of motion. With one more integral and you have solved it. In starting with Newton's 2nd order d.e, you do not avail the benefits of energy conservation.

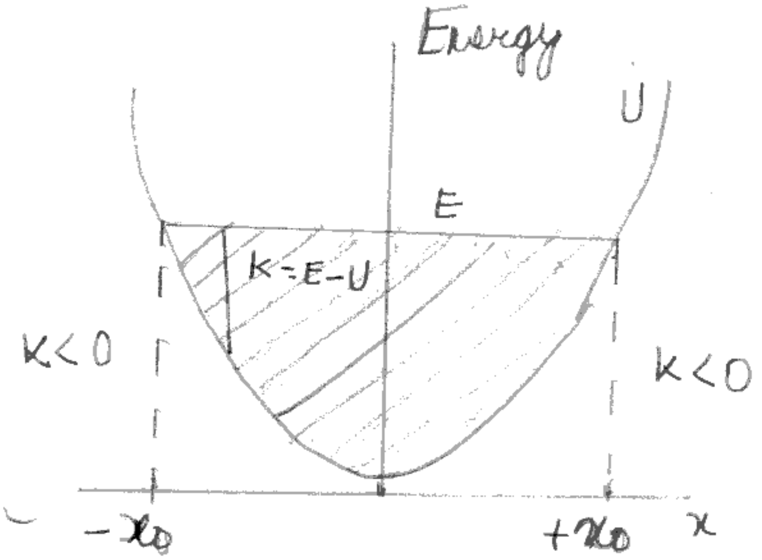
2) ~~th~~ When was this done in the class?

Ans. Short answer: it was not done for the want of time. However, this being such an important and powerful aspect that compares favourably to the much venerated dynamical approach that I could not resist including in my notes.

ENERGY DIAGRAMS

An energy diagram is a plot of total energy and potential energy U as a function of position. Such a simple plot can reveal many key features of the problem without having to solving it. We will look at three situations a) Potentials that lead to bound states b) Only unbound states c) possibility of bound as well as unbounded states.

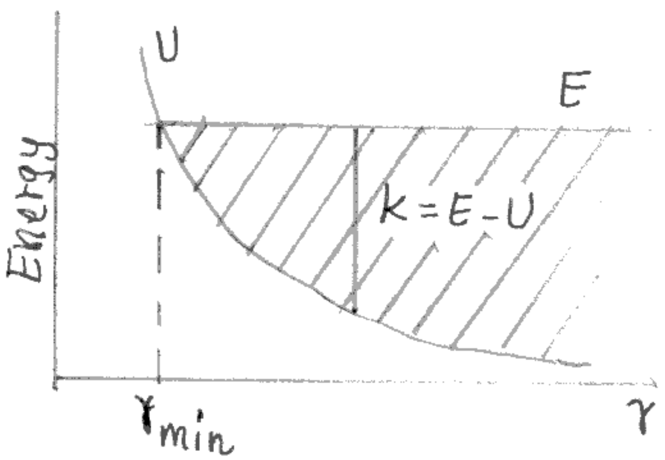
a) HARMONIC OSCILLATOR POTENTIAL: $\boxed{U = \frac{1}{2} k x^2}$



Here is the energy diagram for harmonic oscillator. P.E is a parabola centered at the origin. Because total energy E is a constant for a conservative system E is shown as a horizontal line. Motion is limited to the shaded region where $E \geq U$. This determines the limits of the motion $x = \pm x_0$ which are called turning points. Since $K = E - U$, beyond these turning points, the K.E of the particle would be negative and hence the particle is permanantly confined or bounded within $x = \pm x_0$.

Note that these turning points are determined by the value of the total energy E. Also note that these turning points exist because the potential energy grows indefinitely with distance. It also shows that K.E is zero at the turning points and maximum at the origin and hence the particle accelerating back and forth. Greater the E, further away are the turning points.

b) REPULSIVE INVERSE SQUARE LAW POTENTIAL



$$\vec{F} = \frac{A}{r^2}\hat{r}$$ (A is a positive constant)

$$U = \frac{A}{r} \qquad U(\infty) = 0.$$

Clearly for $r < r_{min}$

$K < 0$

Repulsive inverse square law radial force compels a particle to move along a radial line since for $r < r_{min}$, $K < 0$ there is a distance of closest approach determined by total energy E. Since for $r > r_{min}$ $E > U$ particle accelerates (K.E is increasing) all the way to infinity and hence motion is unbounded (not confined to any region like in harmonic oscillator potential). So if throw the particle towards origin, it rebounds at r_{min} (determined by energy with which we threw) ~~it rebounds at r_{min}~~ and goes back to infinity. Its speed at each point being same during in-bound and out bound journey.

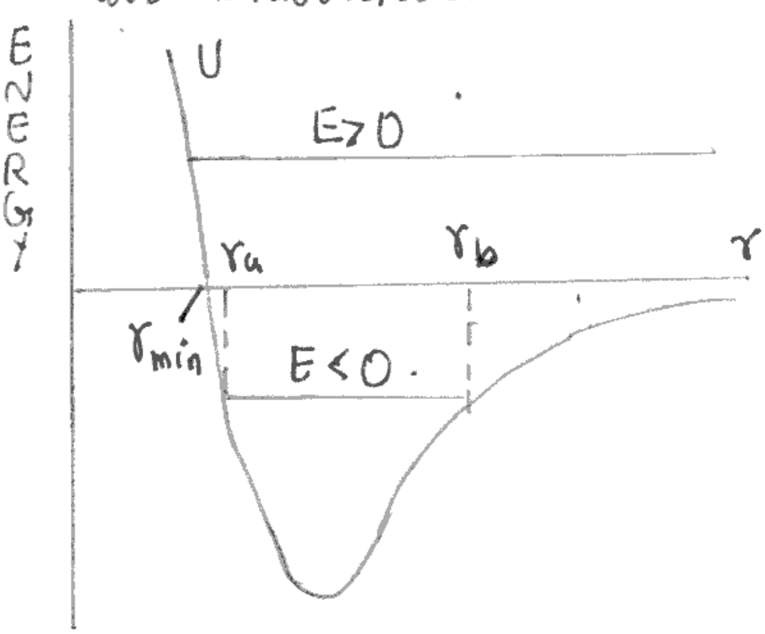
c) POTENTIALS WHICH ALLOW BOUNDED AS WELL AS UNBOUNDED MOTION DEPENDING UPON TOTAL ENERGY

By now it should be pretty clear as to what does it take to have a bounded motion and unbounded motion.

Bounded motion: Potential should have a mimimum so that $\frac{dU}{dx} = 0$.

Unbounded motion Potential should monotonically fall to zero.

To have a possibility of bounded as well as unbounded motion, both these ~~possi~~ features are present. Van der Waals potential shown here is a case in point. For $E > 0$, though there is a fixed distance of closest approach, for $r > r_{MIN}$ K.E is always positive



and the motion is unbounded. However, for $E<0$, K.E is <0 for ~~r~~ $r<r_0$ and $r>r_0$. The motion is clearly bounded. This tells us that when two atoms approach ~~eath~~ each other with $E>0$ (such as collision of two hydrogen atoms in a gaseous state) they will reflect after reaching the distance of closest approach (r_{MIN}) and will never form a molecule. However, if there is some means to loose excess energy to make E negative, then they may form a bound state. The means could be the presence of third atom or a surface. For instance if we insert a piece of platinum in the hydrogen gas, then the hydrogen atoms tightly adhere to the surface of the platinum and if a collision occurs between two atoms at the surface, the excess energy is released to the surface, and the molecule which is not strongly attracted to the surface, leaves. In fact, so much energy is delivered to the platinum that it glows brightly. A third atom can also take away the excess energy, but that is a rare event at low pressures, but it become important at high pressure.

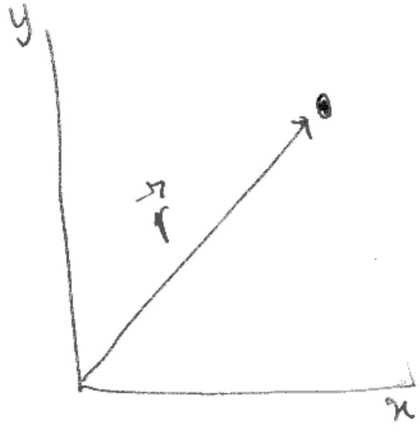
<u>Torque</u>: $\vec{\tau} = \dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$.

Two subtleties

1) internal forces within a collection of particles.
2) Possible acceleration of origin (with respect to which torque and angular momentum are calculated)

Our derivation of $\vec{\tau} = \dfrac{d\vec{L}}{dt}$ is completely general.

A) <u>POINT MASS, FIXED ORIGIN</u>



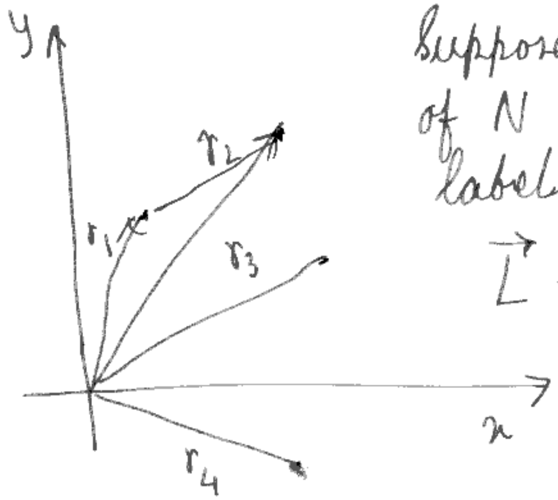
$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times \vec{F} = \vec{\tau}.$$

B) <u>EXTENDED MASS, FIXED ORIGIN</u>

For an extended object, there are internal forces. We will be ~~concern~~ concerned only with central forces.



Suppose we have a collection of N discrete particles labeled by index i.

$$\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i$$

$$\vec{F}_i = \vec{F}_i^{ext} + \vec{F}_i^{int} = \frac{d\vec{P}_i}{dt}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i = \sum_{i=1}^{N}\left[\underbrace{\frac{d\vec{r}_i}{dt} \times \vec{P}_i}_{\to 0} + \vec{r}_i \times \frac{d\vec{P}_i}{dt}\right]$$

$$= \sum_{i=1}^{N} \vec{r}_i \times (F_i^{INT} + F_i^{EXT}).$$

$\sum_{i=1}^{N} \vec{r}_i \times F_i^{INT}$?

$r_1 \times \vec{F}_1 + r_2 \times F_2$

$r_1 + r = r_2$

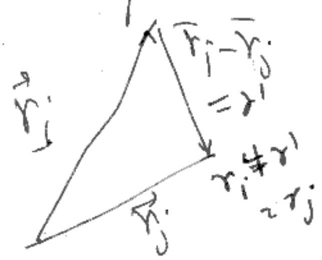
$r_2 = r - \vec{r}_1$

$\vec{r}_1 \neq r - \vec{r}_1$

<u>Zero Torque from internal forces:</u>

Let $\vec{F}_{ij}$ be the force on ith particle due to jth particle.

$$\vec{F}_i^{INT} = \sum_j F_{ij}^{INT}$$



Total internal torque ~~due to~~ on all the particles relative to the chosen origin, is

$$\tau^{INT} \equiv \sum_i \vec{r}_i \times \vec{F}_i^{INT} = \sum_i \sum_j \vec{r}_i \times \left(\vec{F}_{ij}^{INT} \right)$$

If we interchange the indices which were labelled arbitrarily.

$$\tau^{INT} = \sum_j \sum_i \vec{r}_j \times (F_{ji}^{INT}) = -\sum_j \sum_i \vec{r}_j \times (\vec{F}$$

Using Newton's third

Adding the two previous equations

$$2\tau^{INT} = \sum_i \sum_j (r_i - r_j) \times F_{ij}^{INT}$$

But $(\bar{r}_i - \bar{r}_j)$ is parallel to $\vec{F}_{ij}$ by assumption. So the torques cancel in pairs.

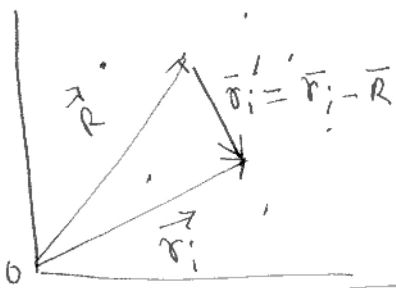
$$\tau^{ext} = \vec{r}_i \times \vec{F}^{ext} = \frac{d\vec{L}}{dt}$$

Thus, if there are no external ~~force~~ torques a rigid body wont spontaneously start rotating. Note that nowhere did we assume that particles are rigidly connected to each other. Eq^n holds even if there is a relative motion among the particles. But in that case it is hard to get a handle on $\vec{L}$ as it does not take $I\omega$ form

ANGULAR MOMENTUM OF A RIGID BODY THAT IS TRANSLATING AS WELL AS ROTATING:

Consider a rigid body as an assembly of large number of particles each of mass m_i and have position vector $\vec{r}_i$ with respect to some fixed inertial origin.

$$\vec{L} = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i$$



This is correct but very boring. Hardly provides any insight about the details of dynamics.

$\vec{r}_i$ = P.V. of i^{th} particle with respect O (fixed).
$\vec{R}$ = P.V of CM of rigid body
$\vec{r}_i{}'$ = P.V of i^{th} particle w.r.t. CM
$\vec{r}_i{}' = \vec{r}_i - \vec{R}$
$\vec{r}_i = \vec{r}_i{}' + \vec{R}$.

Note that $\vec{r}_i = \vec{r}_i{}' + \vec{R}$

Such a decomposition splits the dynamics into $\vec{r}_i{}'$ (motion about CM) and $\vec{R}$ (motion of the CM). This looks interesting.

Upon substitution $\vec{L}$ becomes.

$$L = \sum_i (\vec{r}_i{}' + \vec{R}) \times m_i (\dot{\vec{r}}_i{}' + \dot{\vec{R}})$$

$$= \underbrace{\sum_i \vec{r}_i{}' \times m_i \dot{\vec{r}}_i{}'}_{A} + \underbrace{\sum_i \vec{r}_i{}' \times m_i \dot{\vec{R}}}_{B} + \underbrace{\sum_i \vec{R} \times m_i \dot{\vec{r}}_i{}'}_{C} + \underbrace{\sum_i \vec{R} \times m_i \dot{\vec{R}}}_{D}.$$

(A) $\equiv \sum_i \vec{r}_i{}' \times m_i \dot{\vec{r}}_i{}'$ This is obviously $\vec{L}$ about CM; that is L_{CM}.

(B) $\equiv \sum_i (\vec{r}_i{}') \times m_i \dot{\vec{R}} = \sum_i (\vec{r}_i - \vec{R}) m_i \times \dot{\vec{R}}$ $\boxed{M = \sum_i m_i}$

$$= \sum_i (\underbrace{m_i \vec{r}_i - M\vec{R}}_{\to\, 0 \text{ (Def}^n \text{ of CM)}}) \times \dot{\vec{R}}$$

$$= 0.$$

(C) $\equiv \sum_i \vec{R} \times m_i \dot{\vec{r}}_i{}' = 0$ (In B we proved that $\sum_i m_i \vec{r}_i{}' = 0$, so $\sum_i m_i \dot{\vec{r}}_i{}' = 0$)

(D) $\equiv M\vec{R} \times \dot{\vec{R}}$ $\equiv$ Ang momentum of a rigid body due to translation of CM.

Thus.

$$\vec{L} = \underbrace{\vec{L}_{CM}}_{\vec{L} \text{ in the CM frame (SPIN PART)}} + \underbrace{\vec{R} \times M\dot{\vec{R}}}_{\vec{L} \text{ due to CM motion } \cancel{\text{with}} \text{ w.r.t some fixed origin (ORBITAL PART)}}.$$

$\boxed{L^z_{CM} = (I\omega)_z \text{ for fixed axis rotation}}$

CONSERVATION OF ANGULAR MOMENTUM

CENTRAL FORCES AND KEPLER'S LAW

$\vec{F} = \frac{d\vec{P}}{dt} \Rightarrow$ if $\vec{F} = 0$ $\vec{P}$ = const.

$\vec{\tau} = \frac{d\vec{L}}{dt}$; $\vec{\tau} = 0 \Rightarrow \vec{L}$ is conserved.

$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \vec{F}$ need not be zero for $\vec{\tau} = 0$.

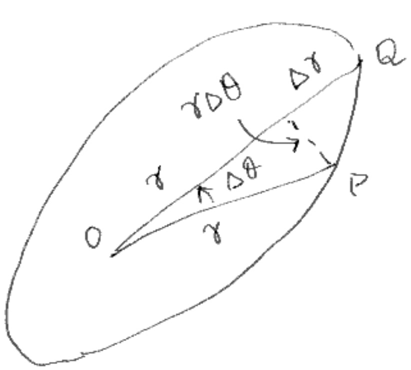
For central (radial) forces : $\vec{F} = f(r)\hat{r}$.

$\vec{\tau} = \vec{r} \times f(r)\hat{r} = 0 \Rightarrow \vec{L}$ is conserved.

If we take direction of $\vec{L} = |\vec{L}|\hat{z}$, conservation means it will always be $\hat{z}$.

Now $\vec{L} = \vec{r} \times \vec{P} \Rightarrow$ the motion is always in x-y plane.

MOTION OF PLANETS : Since gravity is a central force, the motion of planets is confined to the plane. Let us find Areal velocity of a planet going from P to Q.



Area $\Delta OPQ = \frac{1}{2}(r\Delta\theta)(r+\Delta r)$.

$$\Delta A = \frac{1}{2}r^2\Delta\theta + \underbrace{\frac{1}{2}r\Delta\theta\Delta r}$$

2^{nd} order in differential hence $\xrightarrow[\Delta t \to 0]{} 0$.

$$\frac{dA}{dt} = \lim_{\Delta t \to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\omega$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\omega}$$

$$\vec{L} = \vec{r} \times m\dot{\vec{r}} = r\hat{r} \times m(r\dot{\theta}\hat{\theta} + \dot{r}\hat{r}) = mr^2\omega\hat{z}.$$

$\Rightarrow \boxed{\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m}} \Rightarrow$ CONSERVATION OF L AND AREAL VELOCITY ARE CONNECTED.

Thus, Kepler's second law of constancy of Areal velocity ~~is only~~ holds true very generally for all central forces, because conservation of angular momentum is a generic feature of central forces as seen below:

Central force $\Rightarrow F_\theta = m a_\theta = 0$.

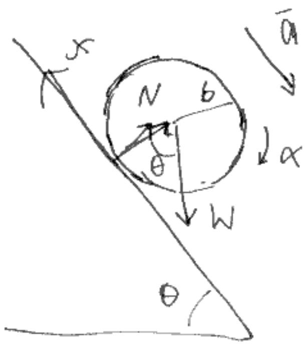
$$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

$$\Rightarrow m(r^2\ddot{\theta} + 2\dot{r}r\dot{\theta}) = 0$$

$$\frac{d}{dt}(\underbrace{mr^2\dot{\theta}}_{L}) = 0$$

$\vec{F} = f(r)\hat{r}$ $\longleftrightarrow$ L = Const $\longleftrightarrow$ $\frac{dA}{dt} = 0 \Rightarrow A$ = const. $\longleftrightarrow$ $\vec{F} = f(r)\hat{r}$

EXAMPLE 6.16: DRUM ROLLING DOWN PLANE



A uniform drum of radius b and mass M rolls w/o slipping down a plane inclined at an angle θ. Find the a. ($I_0 = Mb^2/2$)

Sol: We will solve this problem by taking torque about three different points.

METHOD-1: $W\sin\theta - f = ma$ Translation of CM.

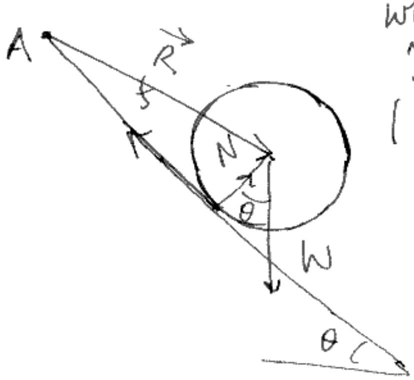
$bf = I_0\alpha$ $\boxed{\text{Torque about CM}}$

$a = b\alpha$ rolling w/o slipping.

Eliminating f and using $I_0 = Mb^2/2$

$$\boxed{a = \frac{2}{3} g\sin\theta}$$

METHOD-2: Let us choose a coordinate system whose origin is A, on the plane. Torque about A is.



$$(\vec{\tau})_{A_z} = \tau_0 + (\vec{R}\times\vec{F})_z$$

Like L, τ also splits into two parts. Here $\vec{R}$ is position vector of CM from A. $\vec{F}$ = Net external force.

$$(\tau_A)_z = \tau_0 + (\vec{R}_\perp + \vec{R}_\parallel)\times(\vec{N}+\vec{W}+\vec{f})$$

$$= -bf + \vec{R}_\perp\times\vec{N} + \vec{R}_\perp\times\vec{W} + \vec{R}_\perp\times\vec{f} + \vec{R}_\parallel\times\vec{N} + \vec{R}_\parallel\times\vec{W} + \vec{R}_\parallel\times\vec{f}$$

$$= \cancel{-bf} + 0 + -bW\sin\theta + \cancel{bf} + R_\parallel N - R_\parallel W\cos\theta + 0$$

$$(\tau_A)_z = -bW\sin\theta$$

$$(L_A)_z = L_{CM} + (\vec{R}\times M\dot{\vec{R}})_z$$

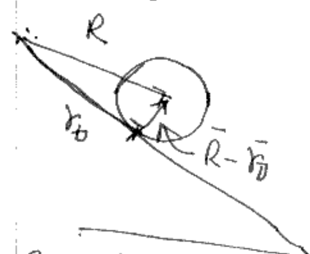
$$= -\frac{1}{2}Mb^2\omega - Mb^2\omega$$

$$= -\frac{3}{2}Mb^2\omega$$

Since $\tau_z = dL_z/dt \Rightarrow bW\sin\theta = \frac{3}{2}Mb^2\dot{\omega}$

$$\Rightarrow \dot{\omega} = \alpha = \frac{2W}{3Mb}\sin\theta \quad \text{or} \quad \boxed{a = b\alpha = \frac{2}{3}g\sin\theta}$$

METHOD 3: Origin at the point of contact. ~~The~~ Since point of contact is accelarating we must use the general formula for torque.



$$\vec{\tau} = \underset{①}{\sum_i (\vec{r}_i - \vec{r}_0)\times\vec{F}^i_{ext}} - \underset{②}{M(\vec{R} - \vec{r}_0)\times\ddot{\vec{r}}_0}$$

① $= -bW\sin\theta$.

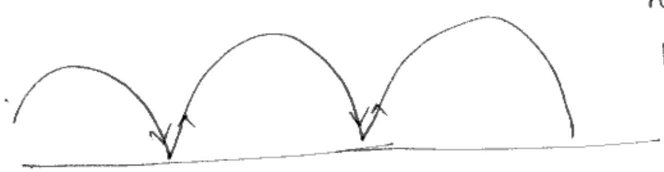
Here the ② term vanishes because cross product vanishes. Velocity of point of contact is downwards just before it touches plane and upwards just after that. Hence $\ddot{\vec{r}}_0$ is facing down normal to incline.

So $(\bar{R} - \bar{r}_D) \times \ddot{\bar{r}}_D = 0$

Pointing up normal to incline

Pointing down normal to incline

Here of course the position vector of origin (point of contact) is $\bar{r}_0$. The fact that the accelaration of point of contact ~~is pointing~~ ($\ddot{r}_0$) is pointing down can be understood from the fact that trajectory of any point on a rolling circle is a cycloid.



Just when the point hits the ground its velocity is pointing dowards and immediately after it, upwards. Thus, only first term contributes

$$\tau = -bW\sin\theta = \left(\frac{Mb^2}{2} + Mb^2\right)\alpha = \frac{3}{2}Mb^2\alpha$$

$$\Rightarrow \boxed{a = \frac{2}{3}g\sin\theta} \quad \text{since } a = b\alpha.$$

The important point to realize here that in general the second term exist. You must not neglect it without knowing why it does not contribute.

METHOD-4: We will now employ energy method and find the speed of rolling drum as it descends through height h. The drum starts at rest.

Translational Work-energy theorem.

$$\int_a^b \vec{F}\cdot d\vec{r} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2 = \frac{1}{2}MV^2$$

$$(W\sin\theta - f)l = \frac{1}{2}MV^2 \quad ① \quad \boxed{l = h/\sin\theta}$$

For the rotational motion

$$\int_{\theta_a}^{\theta_b} \tau_0\, d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$$

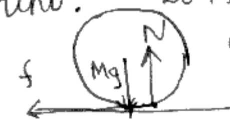
$$fb\theta = \frac{1}{2}I_0\omega^2$$ where θ is the angle through which drum rotates as it translates through l. $l = b\theta$.

$$fl = \frac{1}{2}I_0\omega^2$$

$$fl = \frac{1}{2}\frac{I_0V^2}{b^2} \quad ②$$

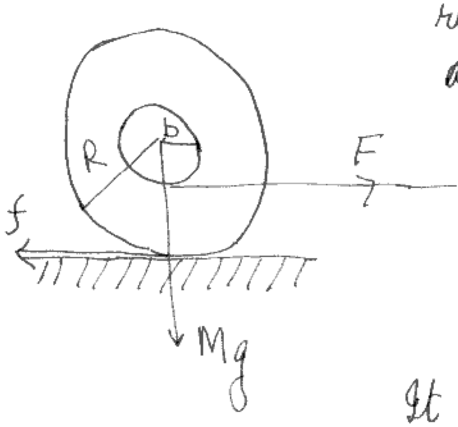
Eliminating f from ① & ② we get $V = \sqrt{\frac{4gh}{3}}$

Interesting thing to note here is that force of friction here is non-dissipative. It decreases translational energy by an amount fl but the torque exerted by friction increases rotational energy by same amount. It is only when a rolling wheel flattens at the bottom that the torque due to N (which doesn't pass from center) decelerates.



6.27 A yo-yo of mass M has an axle of radius b and a spool of radius R. The M.I $= MR^2/2$. Yo-Yo is placed upright on a table and the string is pulled with the horizontal force F. The coefficient of friction between Yo-Yo and table is μ. What is maximum value of F for which Yo-Yo will roll without slipping.

<u>Sol</u>:



Since the Yo-Yo is supposed to roll without slipping, there is a net translational motion as well as rotational motion such that

$\boxed{\begin{array}{l} l = R\theta \\ \text{or } a = R\alpha \end{array}}$ rolling w/o slipping.

It is clear that Yo-Yo will translate to the right as $F > f$ (friction) for translation. Thus

$$F - f = Ma \quad \text{Here } a > 0. \quad (1)$$

There are two torques bF (tending to rotate the Yo-Yo counter clockwise and hence +ve) and fR (tending to rotate the Yo-Yo in clockwise direction and hence -ve). According to translational equation of motion, the Yo-Yo moves to the right. The requirement that it should roll w/o slipping means that the torque which makes it rotate to the right in the clockwise direction ~~that~~ (fR) shall dictate the sign of angular accelaration α. Thus

$$bF - fR = -\frac{MR^2}{2}\alpha = -\frac{MR^2}{2}\left(\frac{a}{R}\right) \quad (2)$$

Solving (1) and (2) we get

$$\boxed{F = \frac{3fR}{2b+R} = \frac{3\mu MgR}{2b+R}}$$

CH. 8. NON INERTIAL FRAMES

→ We have so far formulated Newton's 2^{nd} law in inertial reference frames. All our coordinate measurements referred to a system of axis which were neither linearly accelarating nor rotating.

→ An inertial reference frame is one in which Newton's first law of inertia holds.

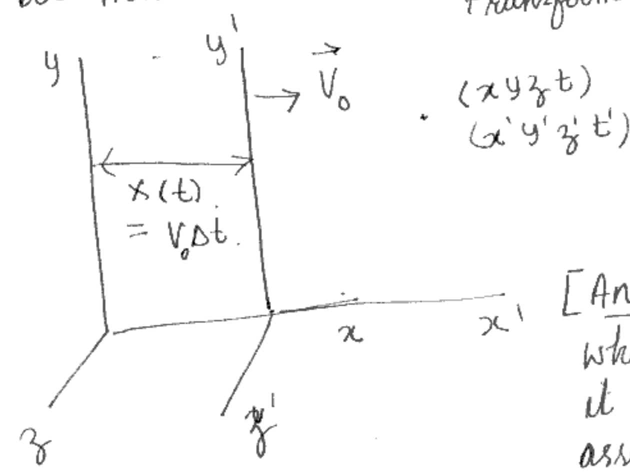
→ We have treated Earth as an inertial reference frame which is technically incorrect. Earth spins about its axis, revolves around sun as well as galactic center. All of these imply centripetal accelaration, however the corrections to g due to its revolution around sun ($a = v^2/r = .006\,m/s^2$) and its own spin ($.03\,ms^{-2}$ or $g/300$ at equator) are very small. However, depending upon the problem at hand this could be important.

→ The purpose of this chapter is to reformulate Newton's 2^{nd} law so that we can address problems of practical interest in non-inertial frames. In fact some problems become much simpler and its physics becomes more transparent when formulated in N.I.F.

GALILEAN TRANSFORMATIONS

→ These relate physical quantities in two different I.F., one of which is moving uniformly w.r.t the other frame.

Let S and S' be two I.F. with S' moving with constant velocity $\vec{v}$ along x' axis. ~~whose~~ At $t=0$, their origins as well as axis coincide. Let (x,y,z,t) and (x',y',z',t') be the coordinate and times of an event as measured in S and S'. Then, Galilean transformations are:



$$x' = x - V_0 t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

[An Aside: Note that when we put $t'=t$, it is a tacit assumption whose validation is subject to empirical verification. Whereas above transformations had ~~emp~~ empirical support at ~~t~~ terrestrial speeds and accuracy of measurements available then. It turns out that the correct relation valid for all possible speeds is given as

$$x' = \gamma(x - vt)\ ;\ y' = y\ ;\ z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right).\quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

These are known as Lorentz transformations and were not discovered in trying to empirically verify Galilean transformation but in trying to fix theoretical inconsisties in the theory of light. It is a consequence of the fact that ~~spe~~ velocity of light in vacuum is a universal constant independent of the velocity of its source. this is at the heart of Einsteins special relativity]

~~Since~~ Writing in a vector form

$$\vec{r}\,' = \vec{r} - \vec{V}_0 t$$
$$\vec{v}\,' = \vec{v} - \vec{V}_0$$
$$\vec{a}\,' = \vec{a}$$
$$m\vec{a}\,' = m\vec{a}$$
$$m\vec{a}\,' = \vec{F}\,' = \vec{F} = m\vec{a}$$

This is the proof that you are free to choose any inertial frame to formulate your 2nd law.

What if S' is accelerating w.r.t. S? Then, say $\vec{A}_0$ is acceleration of S':

$$\vec{a}\,' = \vec{a} - \vec{A}_0$$
$$m\vec{a}\,' = m\vec{a} - m\vec{A}_0$$
$$\vec{F}\,' = \vec{F} - m\vec{A}_0$$

Then in the inertial frame S, the equation of motion of course takes the cannonical form

$$\vec{F} = m\vec{a}$$

where $\vec{F}$ = vector sum of all physical forces such as $m\vec{g}, \vec{N}, \vec{f}, \vec{T}$ etc.

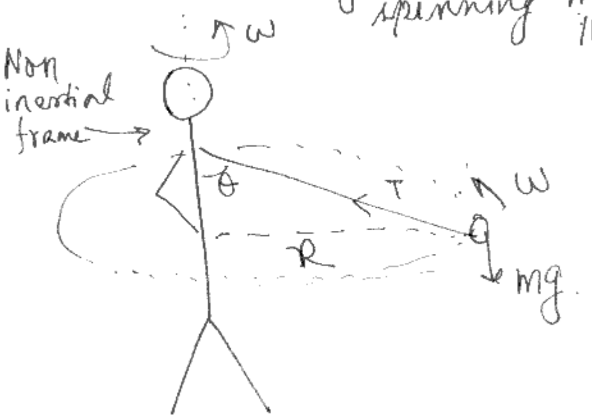
But in the accelarating and hence N.I.F.

$$\vec{F}\,' = m\vec{a}\,'$$
$$\vec{F} - m\vec{A}_0 = m\vec{a}\,'$$

Then, we see that in LHS, over and above ~~sum~~ vector sum of all physical forces, there is an additional term $-m\vec{A}_0$ which does not have a physical origin but is purely a consequence of formulating the problem in a non-inertial frame. It vanishes in the limit $A_0 \to 0$.

Since it does not have a physical origin and is an artifact of accelarating frame, it is aptly called a pseudo force or fictitious force. Proportionality to mass and negative sign (points opposite to $\vec{A}_0$ the accelaration of the frame) are its tell-tale signatures. One very common example of a pseudo-force is centrifugal force (and NOT centripetal).

Example: Mass tied to an almost massless string and whirled in a circle by a spinning man.



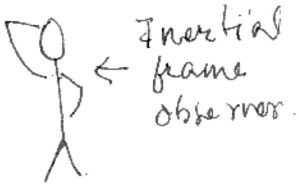
INERTIAL FRAME ~~ASPEC~~ APPROACH:

Here of course mass is going in a circle of radius R and hence needs an agent that provides for necessary centripetal force (radial component of T)

$\hat{y}$: $T\cos\theta = mg$

$\hat{r}$: $-T\sin\theta = -\frac{mv^2}{R}$

$$\boxed{\tan\theta = \frac{v^2}{Rg}}$$



NON-INERTIAL FRAME APPROACH

$$\vec{F}' = m\vec{a}'$$

$$\vec{F} - m\vec{A}_0 = m\vec{a}'$$

Here $\vec{F}$ = vector sum of all physical forces

$\vec{A}_0$ = Acc. of NIF = $\frac{v^2}{R}(-\hat{r})$

$\vec{a}'$ = Acc. observed in NIF.

Since NIF of spinning person has same ω as that of mass m, $\vec{a}' = 0$.

Then $T\cos\theta - mg = 0$

$$\underbrace{-T\sin\theta\,\hat{r}}_{\text{Physical force}} \underbrace{- m\frac{v^2}{R}(-\hat{r})}_{\text{Centrifugal (Pseudo) force}} = 0$$

Again $\boxed{\tan\theta = \frac{v^2}{Rg}}$ Same as before

Not a surprize because if done correctly physics is truly independent of the choice of reference frame.

Centripetal force: It is a REAL force with a physical origin that is has to be provided by a physical agent (f, N, T, $m\vec{g}$, etc) to account for the observed circular motion.

Centrifugal force: It is a fictitious or pseudo force that is invoked to ~~cancel to # some physical force~~ account for the observed accelaration (or lack thereof) ~~of~~ in the non-inertial frame. IT HAS NO PLACE IN ANY ANALYSIS DONE PURELY IN INERTIAL FRAME.

$\boxed{2.29}$ A car is driven on a large rotating platform which rotates with constant angular speed ω. At $t=0$, a driver leaves the origin and follows a radial line with constant speed v_0. The total weight of the car is Mg, and the coefficient of friction between car and platform is μ. a) Find the accelaration of the car as a function of time using polar coordinates. Draw a clear vector diagram showing the components of accelaration at some time $t>0$ b) Find the time t at which the car just starts to skid c) Solve the problem using rotating non-inertial frame of platform.

SOLUTION: Using Inertial reference frame

→ Car rotates with platform with ω and hence needs an agency to provide centripetal accl^n. This is radial component of frictional force f_r.

→ Since car is going with constant speed in radially outward direction $\dot{r} = V = $ const. $\ddot{r} = 0$.

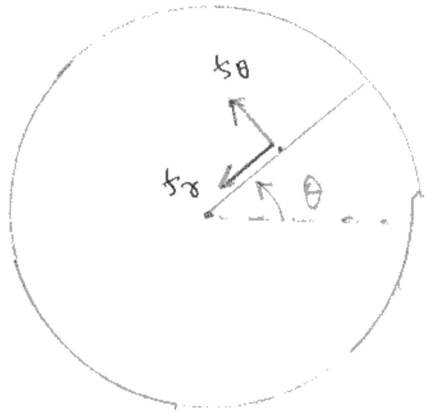
→ Since $\dot{r} \neq 0$, $\omega \neq 0$, ∴ $\dot{r}\omega \neq 0$. Thus there exist a non-zero force in tangential direction. There is nothing other than friction to provide for such a force.

Thus

$$-f_r \hat{r} = m(\underbrace{\ddot{r}}_{0} - r\dot{\theta}^2)\hat{r} \Rightarrow f_r = mr\omega^2.$$

$$f_\theta \hat{\theta} = m(\underbrace{r\ddot{\theta}}_{0} + 2\dot{r}\dot{\theta})\hat{\theta} \qquad \dot{r} = V$$

$$\Rightarrow f_\theta = 2mV\omega.$$



Note that since V is constant, f_θ is always constant, but $f_r = mr\omega^2$ and hence increases linearly with time ($r = Vt$) and distance from the center. The net force on the car (exerted by the platform due to friction) is given in magnitude and direction by the vector sum of f_r and f_θ. Note that the force on the car is reaction force due to the force on the platform exerted by the car, which is equal in magnitude and opposite in direction.

$$\vec{a} = -a_r\hat{r} + a_\theta\hat{\theta}$$

$$|\vec{a}| = (a_r^2 + a_\theta^2)^{1/2}$$

$$a(t) = \left[(vt\omega^2)^2 + (2v\omega)^2\right]^{1/2}$$

car will ~~no~~ not skid until the ~~maximum~~ force f with which it pushes the platform equals the maximum frictional force $f_{max} = \mu Mg = Ma(t)$. Thus

$$(\mu Mg)^2 = M^2 a^2(t)$$

$$\mu^2 g^2 = v^2 t^2 \omega^4 + 4v^2\omega^2$$

$$t = \left[\frac{\mu^2 g^2 - 4v^2\omega^2}{v^2\omega^4}\right]^{1/2}$$

Thus if $4v^2\omega^2 > \mu^2 g^2$ the car will ~~have~~ always skid.

SOLUTION IN NON-INERTIAL FRAM

$$\vec{F}_{rot} = \vec{F}_{IN} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_{rot}.$$

Some observations:

1) In the rotating frame, the car does not rotate. Since it is going with constant speed in radially outward direction $\dot{r} = v_{rot} = v$ and $\ddot{r} = 0$. This there is no radial acceleration in rotating frame ($a_r^{rot} = 0$).

2) Since ω is zero in rotating frame, there is no tangential acceleration in rotating frame ($a_\theta^{rot} = 0$).

Thus LHS of the above equation is:

$$\vec{F}_{rot} = m(\underbrace{a_r^{rot}}_{\to 0}\hat{r} + \underbrace{a_\theta^{rot}}_{\to 0}\hat{\theta}) = 0$$

3) Thus, the three terms on RHS must also "conspire" to give zero. Let us look at them individually and then add components

$$\vec{F}_{IN} = \vec{F}_r^{IN} + \vec{F}_\theta^{IN} = m(\vec{a}_r^{IN} + \vec{a}_\theta^{IN}) \quad ⓐ$$

$$= f_r\hat{r} + f_\theta\hat{\theta}$$

$$m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = mr\omega^2\hat{r} \quad ⓑ$$

$$2m\vec{\omega}\times\vec{v}_{rot} = 2m\omega v\hat{\theta}. \quad ⓒ$$

Thus, adding ⓐ ⓑ and ⓒ and equating LHS = RHS

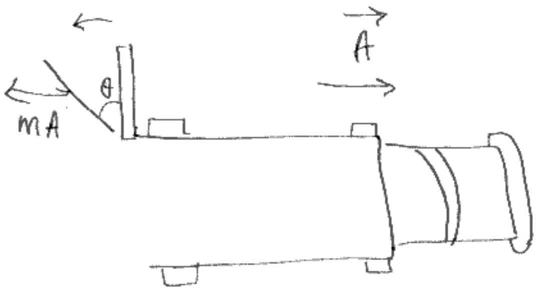
$$0\hat{r} + 0\hat{\theta} = f_r\hat{r} + f_\theta\hat{\theta} - mr\omega^2\hat{r} - 2m\omega v\hat{\theta}.$$

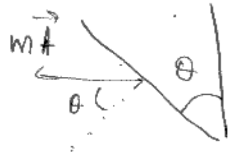
$$0\hat{r} + 0\hat{\theta} = (f_r - mr\omega^2)\hat{r} + (f_\theta - 2m\omega v)\hat{\theta}.$$

$$\Rightarrow f_r = mr\omega^2 \; ; \; f_\theta = 2m\omega v.$$

This is precisely the result we obtained in the inertial frame.

8.2





Since the truck is accelarating, in the frame of the truck there is a torque due to pseudo force whose normal component $mA\cos\theta$ brings about a change in the angular momentum of the door. Equations of motion are:

$$mA\cos\theta\,\frac{l}{2} = I\omega .$$

The door starts at rest and hence initial $\omega = 0$.

$$-N + mA\sin\theta = -m\frac{l}{2}\omega^2 .$$

Work-energy theorem: (pure rotation about pivot) $\int \tau\, d\theta = \frac{1}{2} I\omega^2$

$$\int_0^{90} mA\cos\theta\,\frac{l}{2}\,d\theta = \frac{1}{2} I\omega^2 .$$

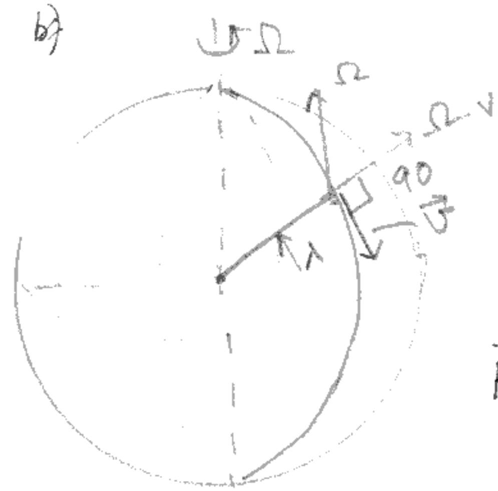
$$\omega^2 = \frac{mAl}{I} .$$

Horizontal component of force when it has swung through 90° : $F_H = m\frac{l}{2}\omega^2 = \frac{m^2l^2A}{2I}$.

8.9

A 400 ton train runs south at a speed 60 miles/hr at a latitude of 60° north.
a) What is the horizontal component of force on the track b) What is the direction of force?

b)



Sol: When you are asked horizontal comp what is meant is the horizontal component of Coriolis force.

$$\vec{F}_{COR} = -2m\vec{\Omega}\times\vec{v}$$
$$= -2m(\vec{\Omega}_V + \vec{\Omega}_H)\times\vec{v}$$
$$= -2m(\vec{\Omega}_V\times\vec{v} + \vec{\Omega}_H\times\vec{v})$$

Now whatever moves on the surface of the earth has its velocity on the plane of the earth. $\vec{\Omega}$ of the earth can be resolved into $\vec{\Omega}_V$ which points in radially outward direction and $\vec{\Omega}_H$ which is on the "plane" of earth. Thus $\vec{\Omega}_H\times\vec{v}$ points in vertical (radial) direction and hence we are not interested in it. $\vec{\Omega}_V\times\vec{v}$ is what leads to horizontal component of Coriolis force.

Thus, $F_H^{COR} = -2m(\vec{\Omega}_v \times \vec{V})$

$|F_H^{cor}| = +2m\Omega \sin\lambda\, V$

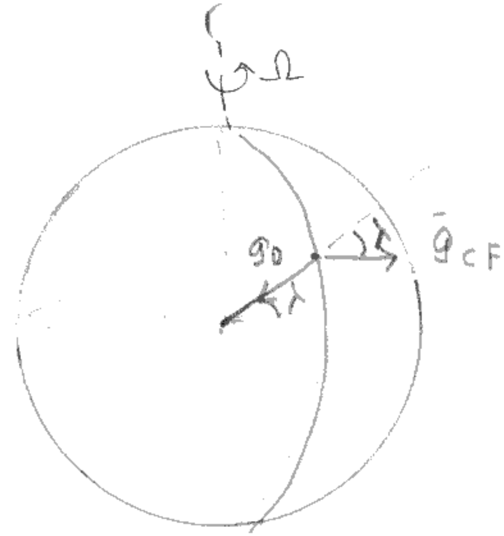
$= \dfrac{2m\Omega V\sqrt{3}}{\cancel{\sqrt{3}}\; 2}$ $\quad M = 4\times 10^5$ kg

$\Omega = \dfrac{2\pi}{24\times 60\times 60}$ rad/s

$V = 60$ miles/hr.

The force on the train is in westward direction and hence the force on the track is ~~eastward~~ also westward.

$\boxed{8.10}$ The accelaration due to gravity measured in a earth bound system is detoned by $\vec{g}$. However due to earth's rotation, g differs from the true accelaration due to gravity g_0. Assuming that the earth is perfectly round, with radius R_e and angular velocity Ω_0, find g as a function of latitude λ. (Assuming that earth is perfectly round is not justified here – the contribution due to polar flattening is comparable to the effect calculated here).



Here,
$\vec{g}_0 = |\vec{g}_0|(-\hat{r})$
$\vec{g}_0$ is the value if earth was not rotating.
$\vec{g}_{CF} = \vec{F}_{CF}/m$
that is correction to $\vec{g}_0$ due to earth's rotation.

Thus $\vec{g}$ as measured on ~~a~~ rotating earth:

$\vec{g} = \vec{g}_0 + \vec{g}_{CF}$ $\qquad |\vec{g}_{CF}| = \cancel{m}\,\Omega^2 R_e \cos\lambda$

Comparing with $(\vec{a}_{rot} = \vec{a}_{in} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}))$ if you are puzzled with a + sign in front of $\vec{g}_{CF}$ then you must note that we have already taken care of direction of $\vec{g}_{CF}$ as it is pointing axially outwards. Thus,

$|\vec{g}| = [\vec{g}_0.\vec{g}_0 + \vec{g}_{CF}.\vec{g}_{CF} + 2\vec{g}_0.\vec{g}_{CF}]^{1/2}$

$= g_0^2\left[1 + \dfrac{(\Omega^2 R_e\cos\lambda)^2}{g_0^2} \cancel{+} - \dfrac{2\Omega^2 R_e\cos^2\lambda}{g_0}\right]^{1/2}$

$= g_0^2[1 + x^2\cos^2\lambda - 2x\cos^2\lambda]^{1/2}$ $\qquad x = \dfrac{\Omega^2 R_e}{g_0}$.

Example 6.11 Analysis of Coriolis force ~~on~~ to calculate the motion of Foucalt pendulum demonstrates rotation of earth beyond doubt.

Consider a pendulum of mass m and frequency $\beta = \sqrt{g/l}$. If we describe the motion of pendulum's bob in a horizontal plane by coordinates r, θ then.

$$r = r_0 \sin \beta t$$

r_0 = amplitude of oscillation

In the absence of Coriolis force there are no tangential forces and θ is constant. Since the bob is moving in a rotating frame, $\vec{F}_{COR} \neq 0$.

$$\vec{F}_{COR} = -2m\Omega \sin\lambda \, \dot{r} \, \hat{\theta}.$$

Hence tangential eqn of motion is $F^{\theta}_{COR} = m a_\theta$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -2m\Omega \sin\lambda \, \dot{r}$$

The simplest solution is found by taking $\ddot{\theta}$

$\dot{\theta}$ = const. $\Rightarrow$ $\dot{\theta} = -\Omega \sin\lambda$.

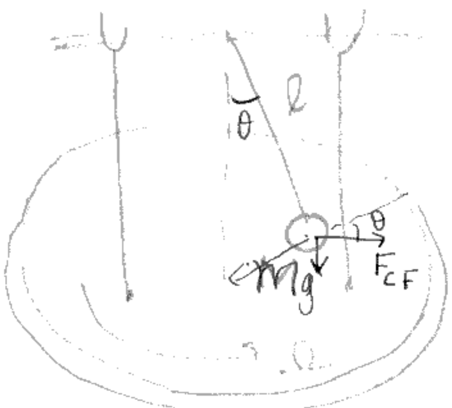
The pendulum precesses uniformly in clockwise due. The time for the plane of oscillation to rotate once is

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\Omega \sin\lambda} = \frac{24h}{\sin\lambda}.$$

at $\lambda = 45^\circ$ $T = 34h$.

The rotation of the plane of oscillation of the pendulum demonstrates the rotation of earth. From an inertial frame one can actually see that plane of oscillation of pendulum remains fixed, but it is the earth beneath which is rotating.

8.12 A pendulum is rigidly fixed to an axle by two ~~of~~ supports so that it can swing only in a plane perpendicular to the axle. The pendulum consists of a mass m attached to a rod of length l. The supports are mounted on a platform which rotates with constant angular velocity Ω. Find the pendulum's frequency assuming that the amplitude is small.



Sol: Note that Coriolis force would tend to move the pendulum out of the plane of oscillation and in clockwise direction. However it cannot succeed because the pendulum is rigidly supported at the pivot. Thus the pendulum is only subject to gravity and centrifugal force due to rotation of platform. Thus equation of motion is:

$$I\ddot{\theta} = \vec{F}_g + \vec{F}_{CF}$$

$$\vec{F}_g = -mg\sin\theta\,\hat{\theta} \qquad |\vec{F}_{CF}| = m\Omega^2 l\sin\theta$$

direction is shown in fig.

EOM is (small θ) $\sin\theta \approx \theta$ $\cos\theta \approx 1$.

$$I\ddot{\theta} = -mgl\theta + m\Omega^2 l^2\theta \underbrace{\cos\theta}_{\approx 1}$$

$$\cancel{m}l^{\cancel{2}}\ddot{\theta} = -(\cancel{m}g\cancel{l} - \cancel{m}\Omega^2 l\cancel{l})\theta$$

$$\boxed{\text{Frequency} = \left(\frac{g}{l} - \Omega^2\right)^{1/2}}$$